BABBITT

A Study of Analytic Functions Based upon the Theory of Potential Functions

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A STUDY OF ANALYTIC FUNCTIONS
BASED UPON
THE THEORY OF POTENTIAL FUNCTIONS

BY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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Historical Note on the Potential.

The function now called the Potential was used by Legendre in 1784, who refers to it when discussing the attraction of a solid of revolution. Legendre however expressly ascribes the introduction of the function to Laplace and quotes from him the theorem connecting the components of attraction with the differential coefficients of the function. M. Bianco in the Rivista di Matematica, 1893, gives quotations from Bist (Institut Paris, 1806) and from Baltzer (Geshichte des Potentials, 1879) showing that Lagrange used the same function in 1777 when discussing the motion of several bodies mutually attracting each other (Academy of Berlin, 1777). The name, Potential, was first used by Green in his Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, published in 1826. Green gave many of the theorems on this function now in use, which have been since associated with the names of others (Gauss, Chasles, Sturm and Thomson) who have discovered them a second time. Gauss also uses the name in article three of his memoir on Forces acting inversely as the square of the distance, Leipzig, 1840, translated in the third volume of Taylor's Scientific Memoirs. The reader may also consult Todhunter's History, Articles 790, 1138, and Thomson and Tait's Treatise on Natural Philosophy, Article 483.

(From Routh's Analytical Statics, Vol. II, p. 20, (Footnote))

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Preface.

The purpose of this thesis is to make a study of analytic functions as based upon potential functions, and thus develop a theory of analytic functions from the standpoint of the physicist.

The first chapter of the thesis deals with the general properties of potential functions. The second chapter is devoted to the study of conjugate functions and functions which are holomorphic in a given region. The subject of conjugate functions is taken up from a point of view different from that generally taken. The definition of conjugate functions is given in concepts of physics rather than of mathematics. The general properties of conjugate functions are considered from the point of view of the definition introduced. The third chapter deals with the subject of singularities and fundamental theorems of analytic functions and that property of potential functions which, because of its analogy with a similar property possessed by analytic functions, we call "potential continuation". At the end a definition of an analytic function is given. Throughout the thesis we have confined ourselves to the consideration of single-valued functions. No attempt is made to give a complete development of analytic functions; but the most important points in the general theory are considered.

In concluding, I wish to express my gratitude to Professor Jacob Kunz of the Department of Physics for his kind advice in matters pertaining to the physics side of this thesis,

and especially to express my indebtedness to Professor E. J. Townsend, Head of the Department of Mathematics, for his support and encouragement without which this thesis would have been impossible.

University of Illinois,

Urbana,

May, 1915.

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CHAPTER I.

Introduction.

General Properties of the Potential.

1. Definition of Potential.[1]

The Potential at any point, due to any attracting or repelling body, or distribution of matter, is the mutual potential energy[2] between it and a unit of matter placed at that point. According to the convention for zero now adopted,[3] when two bodies attract mutually, their mutual potential energy is negative. In the case of gravitation, however, to avoid defining the potential as a negative quantity, it is convenient to change the sign. Then, we define the gravitation potential at any point, due to any mass, as the quantity of work required to remove a unit of matter by any path from that point to an infinite distance.

Now consider two particles, P and Q, whose masses are unity and m, and the coordinates x, y, z and a, b, c respectively. If r be their distance

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1. The introduction of the notion of Potential in mathematics has its origin from a remark made by Lagrange according to which the components of a force acting according to Newton's law of attraction, may be represented as partial derivatives of a single function in the direction in which the components aim. This single function has received the name of "Potential Function". (Plemelj, Potentialtheoretische Untersuchungen, p. 1) In this connection the reader is referred to § 2 of Plemelj's excellent work here quoted. The potential is there defined in a new and interesting way.

2. See: Thomson and Tait, Natural Philosophy, Vol.II, §484,p.28

3. Thomson and Tait. Loc. cit. Vol. II, § 547, p. 93.

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 .$$

The components of the mutual attraction are

$$X = m\,\frac{x-a}{r^3}\,, \qquad Y = m\,\frac{y-b}{r^3}\,, \qquad Z = m\,\frac{z-c}{r^3}\,,$$

and therefore the work required to remove P to infinity is

$$m \int_{x,y,z}^{\infty} \frac{(x-a)dx+(y-b)dy+(z-c)dz}{r^3} \,. \tag{1}$$

Denoting the work done on the mass m in bringing it from the point P(x,y,z) to infinity by V.m, we have

$$V = \int_{x,y,z}^{\infty} \frac{(x-a)dx+(y-b)dy+(z-c)dz}{r^3} \equiv \int_{x,y,z}^{\infty} (Xdx+Ydy+Zdz)^{1} \tag{1'}$$

giving a mathematical expression for the potential at the point P(x,y,z).

The foregoing definition of the potential may be called the analytical definition. This definition, as can be readily seen from our discussion, rests upon the principle of work. We shall now proceed to give the so-called geometrical definition of the potential.

From the expression for r^2 (p.1) and from (1) it follows, that

$$m \int_{x,y,z}^{\infty} \frac{(x-a)dx+(y-b)dy+(z-c)dz}{r^3} = m \int_{x,y,z}^{\infty} \frac{dr}{r^2}$$

which, since the upper limit is $r = \infty$, is equal to $\frac{m}{r}$.

Thus the potential at P(x,y,z) due to the mass m is $\frac{m}{r}$.

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1. Compare: Jeans, Electricity and Magnetism. (Second Ed.) p. 27, (6).

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If there be more than one fixed particle m, the same investigations will lead us to conclude that the potential at P(x,y,z) is $\sum \frac{m}{r}$, which is the common definition of the potential.

Now, if the particles form a continuous mass, (Fig.1), and if we denote by Δm a mass element, then the potential at any point P(x,y,z) due to the mass Δm at Q(a,b,c) is

$$\Delta V = \frac{\Delta m}{r} ,$$

where r is the distance of the point P(x,y,z) from Q(a,b,c).

The <u>whole</u> <u>potential</u> at P(x,y,z) is the sum of that due to all parts of the attracting body, or the volume integral

$$V = \iiint_C \frac{dm}{r} ,$$

the limits depending on the boundaries of the mass.

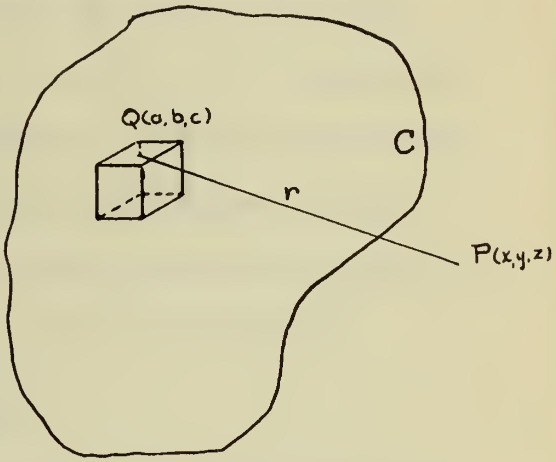

Fig. 1.

Denoting now by ρ the average density of the element whose mass is Δm, we have, (in rectangular coordinates)

$$\Delta m = \rho \Delta a . \Delta b . \Delta c ,$$

and since $r = \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$, it follows that

$$V = \iiint_C \frac{dm}{r} \equiv \iiint_C \frac{\rho . da . db . dc}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}} \qquad (2)$$ [1]

As the position of the point P changes continually the value

1. G. Green introduced for this the name "Potential Function", and C.F.Gauss the name "Potential". (Encyclop. der Math. Wiss., Band II, Heft 4, p. 466)

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of the expression under the integral signs in (2) changes continually with a definite rate of change, and for every point x,y,z, V has a single definite value. It is accordingly a uniform function of the point P(x,y,z).

If V be the potential at any point P(x,y,z), it is evident from the way in which its value has been obtained, that the components of attraction on unit of mass at P are

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z}. \tag{3}$$ [1]

It appears from this proposition that, when the potential V of a body fixed in space is given, its resolved attractions at any point P can be found by simply differentiating the potential with regard to the coordinates of that point. Moreover, we could have considered the above three equations jointly as the definition of the Potential.[2]

2. Potential and Potential Function.[3]

As we have already remarked (see footnote of p.3) what we have called "Potential" was called by Green "Potential Function". Some authors use the term "Potential function"[4], but most of the authors of treatises on mathematical physics use the term "Potential"[5] instead. Even in mathematical treatises on the potential theory, the term Potential is being

1. Compare: Routh, Analytical Statics, v.II,§41, p.21, In this connection the reader may find it profitable and interesting to read §42 where reasons for using the potential are given.
2. See: Jacob Kunz. Theoretische Physik., p. 76.
3. For the graphical representation of a potential the reader is referred to G.Holzmüller's Ingenieur Mathematik, Teil II, Das Potential. pp. 24-28.
4. For instance B.Peirce, (Newtonian Pot.), G.A.Webster, (Electr. & Magn.), etc..
5. Thomson & Tait (Natural Phil.); Buchholz (Mechanisches Potential); Jeans (Electr. & Magnetism) etc., etc.

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used more and more. Plemelj, for example, in his researches on the Potential Theory, a work which has been crowned by the Jablonovsky Society of Leipzig, gives one definition for the "Potential function" and"Potential".

The distinction between the terms "Potential function" and "Potential" had been proposed by Clausius, but had never been carried out[1]. The term "Potential" is used when we consider the attraction at a point, while "Potential function" is used in connection with the attraction of a region. We shall use throughout this treatment the terms "Potential" and "Potential function" indiscriminately.

3. Some Properties of the Potential.[2]

It is beyond the scope of this thesis to present here the derivations even of the most important properties of the Potential. Most of the proofs are rather long, and the reader is referred for them to treatises on the potential theory and on mathematical physics. We will merely give here a resumé of some of the properties of the potential which will be made use of later, leaving the discussion of the other properties for a later section.

1) It is seen at once from (3), § 1, that if the Potential at a certain point exists, by virtue of its existence it possesses partial derivatives of the first order.

2) The value of the potential and the component in any direction of the attraction at the point P, are always

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1. Lectures on the Potential Theory by Prof. Voigt given at the Univ. of Gottingen, pp. 18-19. Courtesy of Prof. E. J. Townsend, of the Univ. of Illinois.
2. Other properties will be considered in § 6. A.B.

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finite functions of the space coordinates, whether P is inside, outside, or at the surface of the attracting masses.[1]

3) By differentiating V at any point in any direction we may find the always finite components in that direction of the attraction at the point.[2]

4) It now follows that $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$ are everywhere finite and continuous and that, in consequence of this, the potential is everywhere continuous as well as finite.[3]

5) The values of $\frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial y^2}, \frac{\partial^2 V}{\partial z^2}$ are finite whenever P is within or without the attracting mass, if the derivatives of the density are finite.[4]

6) It is proved that the second derivatives of the potential are finite at all points on the surface of attracting matter where the curvature is finite, but that the normal derivatives generally change their values abruptly whenever the point P crosses a surface at which ρ is discontinuous, as at the surface of the attracting masses. The fact, however, that this last is true in the special case of a homogeneous spherical shell suffices to show that we can not expect all the second derivatives of V to have definite values at the boundaries at attracting bodies.[5]

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1. See: B. Peirce's "Newtonian Potential Function" § 22.
2. B. Peirce. Loc. cit. § 20.
3. B. Peirce. Loc. cit. § 28. Also <u>Webster</u>, Electricity and Magnetism, § 76, pp. 148-151. (Continuity and finiteness of first derivative proved in two ways).
4. B. Peirce. Loc. cit. § 29.
5. B. Peirce. Loc. cit. § 30.

4. Poisson's and Laplace's Equations.

In treatises on the potential theory[1] and likewise in treatises on mathematical physics[2] it is shown that when the point P is situated within the attracting mass, the potential satisfies Poisson's differential equation, namely

$$\Delta V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho\ , \quad ^{3} \tag{1}$$

where ρ is the density (finite) at P. The potential due to any conceivable distribution of matter must be such that at all points within the attracting mass this equation shall be satisfied.

Outside the attracting mass, where $\rho = 0$, Poisson's equation becomes the well known Laplace's Equation, namely

$$\Delta V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2}$$

This equation is by some authors called "the potential equation" and by others "the equation of continuity".

If the conditions, however, are such that the attractive force acts only in a plane, which may be taken for convenience as the XY-plane, then the third component of the force becomes constant and $\frac{\partial^2 V}{\partial z^2}$ vanishes. Consequently, for two dimensions Laplace's equation becomes

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1. B. Peirce. Loc. cit. § 35., particularly pp. 61-62.
2. A. G. Webster. Electricity and Magnetism. § 77.
3. For the operator $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$ the symbols Δ, Δ_2, $-\nabla^2$, ∇^2, δ, and $\bar{\nabla}^2$ have been used by different authors. We will use throughout the first one of these symbols. A.B.

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$$\Delta' V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \ .^{1} \qquad (3)$$

Of the equations here considered for our purpose, far the most important one is equation (3), i.e. Laplace's Equation for two dimensions, and we shall concern ourselves mainly with this equation.

5. Kinds of Potential.

A function $V(x,y,z)$ which satisfies Laplace's Equation for three dimensions, is called a Newtonian Potential Function. A function that satisfies Laplace's Equation is often called a harmonic function[2]. Such a function is a potential function, but every potential function is not on the other hand an harmonic function since, when the points at which the potential is taken lie within the attractive mass, the potential function must satisfy Poisson's Equation, not Laplace's Equation.

When the points at which the function in question is considered lie outside of the attractive mass it may be called either a potential function or a harmonic function. In what follows, we shall speak of the function as a potential function as there can be no occasion for misunderstanding.

It can easily be shown that $\frac{1}{r} = \log \frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$ satisfies Laplace's Equation for three dimensions, but does

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1. The symbol Δ' will be used for the operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. Equation (3) shows that the streaming is steady. (F.Klein. Riemann's Theory of Functions, p. 2. English translation by F. Hardcastle)
2. In this connection the reader may find it interesting to see the Proceedings of American Acad. of Arts & Sciences., Vol. 41, No. 26 (1906), p. 577.

not satisfy the same equation for two dimensions. In the latter case however, the logarithm of the distance, i.e. $\log r \equiv \sqrt{(x-a)^2+(y-b)^2}$ satisfies Laplace's equation for two dimensions.[1] We are now led to the following

Definition. A function which satisfies Laplace's equation for two dimensions is said to be a logarithmic potential function, or simply a logarithmic potential.[2]

There are essentially two kinds, or rather two types of potentials. 1) those due to attraction, and 2) those due to flow (heat, electricity, etc.). In connection with the first type we have the newtonian potential, magnetic potential, electric potential, electrostatic potential, electromagnetic potential, vector potential. In connection with the second type we have the velocity potential of various kinds (flow of heat, of fluids, etc.).

If V is such a function of space coordinates x,y,z that the components of the velocity are

$$-\frac{\partial V}{\partial x},\ -\frac{\partial V}{\partial y},\ -\frac{\partial V}{\partial z},$$

then V is called a velocity-potential.[3]

We should notice here that we could have considered all potentials as belonging to either of the following two

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1. See: H. Bucholz. Mechanisches Potential. pp. 78-79.
2. The name "Logarithmic Potential" was first introduced by C. Newman in 1861. See: Borchardt's Journal, vol. 59, p. 335.
For the geometric interpretation of the Logarithmic potential the reader is referred to H. Bucholz, loc. cit. pp.113-114.
3. For discussion of velocity-potentials see Lamb's Hydrodynamics. Third Edition, Chapters II and III.

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types: 1) logarithmic potential, and 2) velocity potential. Indeed, we could define a potential as a function either of force or of acceleration or of velocity, so that either of them is a derivative of the potential.[1]

6. Characteristics of the Potential.[2]

The potential possesses the following characteristic properties through which it is completely determined.

1. The potential is everywhere finite and continuous.

2. Its first partial derivatives are finite and continuous.

3. Its second derivatives are finite.[3]

4. For points within the attracting mass it satisfies Poisson's Differential equation:

$$\Delta V = -4\pi\rho \ .$$

5. Outside the attracting mass it satisfies Laplace's differential equation:

$$\Delta V = 0 \ .$$

6. It vanishes at infinity.[4]

7. It is totally differentiable.[5]

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1. This definition of Potential has been suggested to me by Professor J. Kunz of the University of Illinois.
2. See: § 3.
3. See in this connection § 3, particularly p. 6.
4. B. Peirce. Newtonian Potential Function. § 26.
5. See: J. Pierpont's The Theory of Functions of Real Variables, vol. 1, § 423 for definition of a totally differentiable function and § 425 for the theorem upon which (7) rests.

CHAPTER II.

Conjugate Functions. Functions Holomorphic in a Region.

1. Definition of Conjugate Functions. Condition for Orthogonality.

A surface, at every point of which the potential V has the same value is called an equipotential or level surface.[1] Thus $V = f(x,y,z) = c$ may be regarded as the general equation of an equipotential surface. The orthogonal trajectories of the family of equipotential surfaces $V = c$ are called "lines of force".[2] It should be noted here that we concern ourselves not with the surfaces themselves, but with their projections upon a plane which for convenience we take as the complex plane. The reader should therefore constantly bear in mind that in all our succeeding discussions the equipotential lines[3] as well as the lines of force lie in the complex plane.

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1. "Equipotential surfaces" are called "level surfaces" by C. Maclaurin, and "surfaces de niveau" by A. Clairaut. (Encycl. d. Math. Wiss. Band II, Heft 4, p. 476).
   The name "surface of equilibrium" is also used. (Thomson and Tait, Natural Philosophie, §487, Routh, Analytical Statics, vol. II, p. 24)
   The German name for "equipotential surface" is "Niveaufläche" or "Aequipotentialfläche". (J. Kunz, Theoretische Physik. p. 69)
2. While the terms "equipotential surface" and "lines of force" are generally employed in connection with phenomena taking place in an electric field or in a field of a gravitational force, we will agree to use these terms in connection with all phenomena which arise in connection with an electric, magnetic, gravitational field, or in connection with the flow of an incompressible fluid.
3. i.e. the traces of the equipotential surfaces upon the Complex plane.

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With these preliminary definitions before us, we are now in position to define conjugate functions.

Definition. Two functions $U(x,y)$ and $V(x,y)$ are called conjugate functions if they satisfy the following conditions:

1. Either U or V may be taken as a potential function the lines of equipotential for the one function then becoming the lines of force for the other function and vice versa.

2. The intensity[1] of the two superimposed fields of force shall be the same in magnitude at each point.

It is evident that the whole system of equipotential lines is cut orthogonally by the system of lines of force[2], and we shall now derive the condition of orthogonality.

Let $U(x,y) = c_1$ and $V(x,y) = c_2$, where c_1 and c_2 are constants, be respectively the equations of the equipotential lines and of the lines of force. From the relation $u(x,y) = c_1$ we have

$$\text{Slope of } U \equiv \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}.$$

In order that the curve given by $V(x,y) = c_2$ be orthogonal to the system $U(x,y) = c_1$ the slope of $v = c_2$ at points of intersection with the curve $u = c_1$ must be the negative reciprocal of the slope of $u = c_1$ at these points. From the relation $V(x,y) = c_2$ we have

$$\frac{dy}{dx}_{(\text{along } v=c_2)} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

1. For definition of Intensity see Jeans Electricity & Magnitism, Second Edition, p. 26.
2. Thomson & Tait. Natural Philosophy, vol.II, (second ed.) § 47.

and hence we have as the condition of orthogonality

$$-\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} = \frac{\dfrac{\partial v}{\partial y}}{\dfrac{\partial v}{\partial x}}$$

which may be written in the form in which is is generally given

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0 .$$

2. Examples of Conjugate Functions.

We shall now consider some examples of conjugate functions and we will show how the curves of the lines of equipotential and the curves of the corresponding lines of force may be plotted.

Example I. Given the functions

$$u \equiv x + a , \quad \text{and} \quad v \equiv y + b .$$

That these two functions are conjugate can be seen at once. Indeed, either of them may be considered as a potential function (Cf.§6,I). Furthermore, if $x + a = c_1$, a constant, be considered as the equation of the lines of equipotential, then $y + b = c_2$, a constant, will represent the equation of the corresponding lines of force. Let us plot the system of curves given by

$$x + a = c_1$$

or what is the same

$$x = c_1 - a \equiv c_1' \quad \text{a constant.}$$

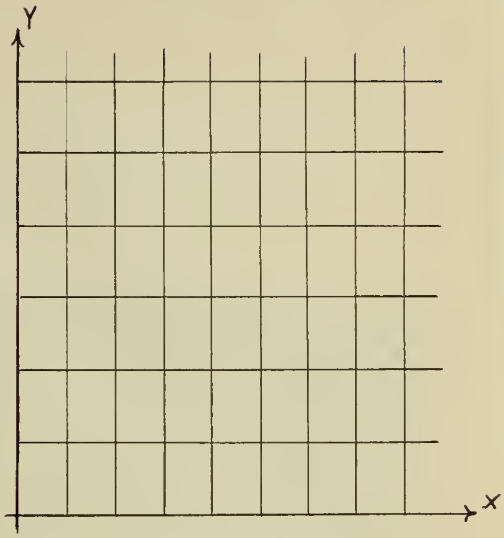

Fig. 2.

As is well known from Analytic Geometry, $x = c'$ represents a system of lines parallel to the Y-axis. (Fig. 2). The system of curves given by

$$y+b = c_2, \quad \text{or} \quad y = c_2-b \equiv c'',$$

is merely a system of lines parallel to the X-axis (Fig. 2). Apparently, the system of curves given by

$$x + a = c_1 \quad \text{and by} \quad y + b = c_2$$

cut everywhere orthogonally.

<u>Note</u>. The fact that the two systems of curves are given by the equation $u = c_1$ and the other by its conjugate function facilitates the ease with which either may be constructed when one of them is given. All that one has to do is to construct a second system everywhere orthogonal to the first.

Example II. Given $u \equiv x^2 - y^2$ and $v \equiv 2xy$.

Either one of the given functions may be regarded as a potential function, since either satisfied Laplace's Equation $\Delta'V = 0$ and since all the other characteristic properties[1] through which a potential is completely determined are here satisfied. The condition of orthogonality is likewise satisfied by the system of curves given by

$$u \equiv x^2 - y^2 = c_1 \quad \text{and} \quad v \equiv 2xy = c_2$$

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1. See § 6, Chapter I.

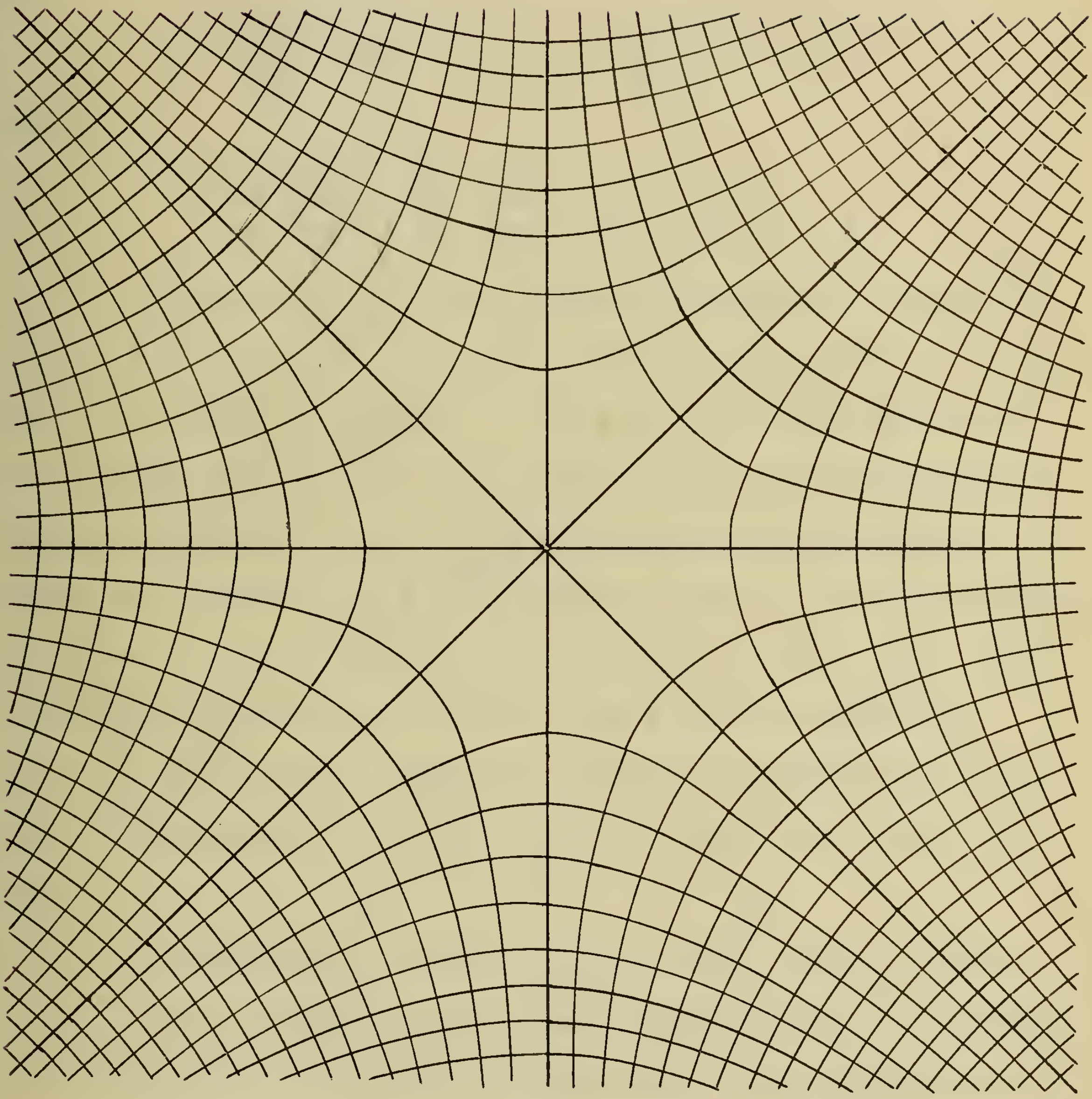

Fig. 3.

For

$$\frac{\partial u}{\partial x} \equiv 2x, \qquad \text{and} \qquad \frac{\partial u}{\partial y} \equiv -2y\ ;$$

Also

$$\frac{\partial v}{\partial x} \equiv 2y, \qquad \text{and} \qquad \frac{\partial v}{\partial y} \equiv 2x\ .$$

Hence

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \equiv 4xy - 4xy = 0\ .$$

In plotting the two surfaces of curves, since we know that u and v are conjugate functions, we make use of note made at the end of Example I. We then plot the system of curves given by $x^2 - y^2 = c_1$, which, as is well known from analytic geometry, is a system of equilateral hyperbolas having the lines $y = \pm x$ as the asymptotes. We then construct a system of curves everywhere orthogonal to this one. (The orthogonal system is, as may be seen from its equation, $2xy = c_2$, a system of equilateral hyperbolas having the two axes as asymptotes). The two sets of curves are shown in Fig. 3.

Example III. Given $u \equiv x^3 - 3xy^2$ and $v = 3x^2y - y^3$.

That either of the given functions may be regarded as a potential is seen at once when upon differentiation and substitution into Laplace's Equation $\Delta' V = 0$, and also applying the characteristic properties[1] for a potential.

To show that the condition of orthogonality is satisfied, we have

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1. See: § 6, Chapter I.

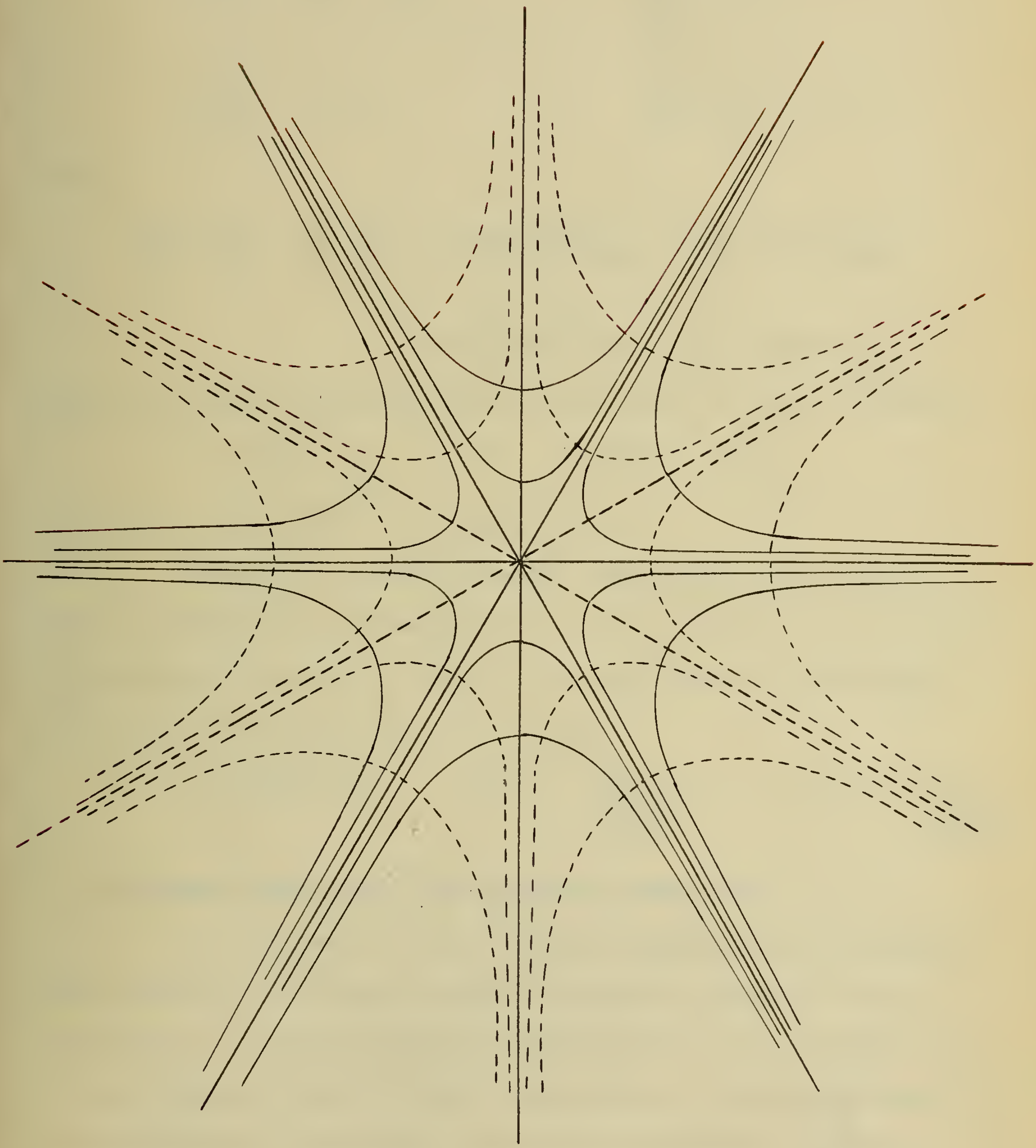

Fig. 4.

$$u \equiv x^3 - 3xy^2 = c_1 \quad \text{and} \quad v \equiv 3x^2y - y^3 = c \ .$$

$$\frac{\partial u}{\partial x} \equiv 3x^2 - 3y^2 \ ; \qquad \frac{\partial v}{\partial x} \equiv 6xy \ ;$$

$$\frac{\partial u}{\partial y} \equiv - 6xy \ ; \qquad \frac{\partial v}{\partial y} \equiv 3x^2 - 3y^2 \ .$$

Hence

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = (3x^2-3y^2)6xy + (3x^2-3y^2)(-6xy)$$

$$= (3x^2-3y^2)6xy - (3x^2-3y^2)6xy = 0 \ .$$

The diagramatic representation of the sets of curves given by

$$x^3 - 3xy^2 = c_1 \quad \text{and} \quad 3x^2y - y^3 = c_2$$

is shown in Fig. 4.[1]

Remark. If two conjugate functions were given in polar coordinates, for example the two functions $r_1 r \cos(\theta_1+\theta)$, $r_1 r \sin(\theta_1+\theta)$, then they would be transformed to cartesian coordinates by means of the relations

$$r = \sqrt{x^2 + y^2} \ , \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \ , \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \ .$$

3. Conjugate Functions and Laplace's Equation.

It follows at once from our definition of conjugate functions (§1) that they satisfy Laplace's equation for two dimensions. For, let $u(x,y)$ and $v(x,y)$ be two conjugate functions. Since either of them, according to our definition, may be regarded as a potential, therefore either one satisfies Laplace's equation for two dimensions, and we have

1. See Klein's "Riemann's Theory of Algebraic Functions"; English translation by F. Hardcastle (1893) p. 3.

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 , \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Illustrative Example. Given two conjugate functions $u = e^x \cos y$ and $v = e^x \sin y$. To show that u and v satisfy Laplace's Equation.

From $u = e^x \cos y$, by differentiating with respect to x, we get $\frac{\partial u}{\partial x} = e^x \cos y$ and differentiating a second time with respect to x, we have $\frac{\partial^2 u}{\partial x^2} = e^x \cos y$.

Now differentiating $u = e^x \cos y$ twice with respect to y we get $\frac{\partial^2 u}{\partial y^2} = - e^x . \cos y$. Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0 .$$

Similarly, by differentiating $v = e^x \sin y$ first twice with respect to x and then twice with respect to y, we obtain $\frac{\partial^2 v}{\partial x^2} = e^x \sin y$ and $\frac{\partial^2 v}{\partial y^2} = - e^x \sin y$. Whence

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x \sin y - e^x \sin y = 0 .$$

4. Conjugate Functions satisfy the Cauchy-Riemann Differential Equations. A necessary and sufficient condition that two functions be conjugate.

It can now be shown that two conjugate functions $u(x,y)$ and $v(x,y)$ satisfy the differential equations, known as the Cauchy-Riemann differential equations.

Indeed, if u(x,y) and v(x,y) are conjugate functions it follows at once from our definition of conjugate functions (§1) and from the characteristic properties of potentials[1], that

(1) $$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 .$$

(2) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ; \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 .$$

(3) $$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \text{ ; } \qquad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0$$
$$(\text{along } u = c_1) \qquad\qquad (\text{along } v = c_2)$$

From (1) we have

$$\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial v}{\partial y}} = - \frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial v}{\partial x}} ,$$

or since each differential quotient here is in general a function of x and y, we may put it equal to an arbitrary function of x and y, say K(x,y) and we have

$$\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial v}{\partial y}} = - \frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial v}{\partial x}} = K(x,y) ,$$

which may be rewritten in the form

$$\frac{\partial u}{\partial x} = K(x,y)\frac{\partial v}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$\frac{\partial u}{\partial y} = - K(x,y)\frac{\partial v}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Differentiating the first equation of (4) with respect to y and the second with respect to x, we obtain

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1. § 6, Chapter I.

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial K(x,y)}{\partial y}\frac{\partial v}{\partial y} + K(x,y)\frac{\partial^2 v}{\partial y^2}, \tag{5}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial K(x,y)}{\partial x}\frac{\partial v}{\partial x} - K(x,y)\frac{\partial^2 v}{\partial x^2} \tag{6}$$

Subtracting (6) from (5), and since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ we have

$$\frac{\partial K(x,y)}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial K(x,y)}{\partial x}\frac{\partial v}{\partial x} = 0 \tag{7}$$

Now, differentiating the first equation of (4) with respect to x and the second with respect to y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial K(x,y)}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x \partial y}K(x,y) \tag{8}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial K(x,y)}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial y \partial x}K(x,y) \tag{9}$$

Adding (8) and (9), and since $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$, we have

$$\frac{\partial K(x,y)}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial K(x,y)}{\partial y}\frac{\partial v}{\partial x} = 0 \tag{10}$$

Considering now (7) and (10) as two homogeneous linear equations, the necessary and sufficient condition that they be consistent[1] gives us

$$\begin{vmatrix} \frac{\partial k}{\partial x} & -\frac{\partial k}{\partial y} \\ \frac{\partial k}{\partial y} & \frac{\partial k}{\partial x} \end{vmatrix} = 0, \quad k \equiv K(x,y)$$

or

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1. See Weld's Theory of Determinants. (New York, 1893), p. 92, (italics).

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$$\left(\frac{\partial k}{\partial x}\right)^2 + \left(\frac{\partial k}{\partial y}\right)^2 = 0 .$$

Hence we have $\frac{\partial k}{\partial x} = 0$, and $\frac{\partial k}{\partial y} = 0$, and it follows at once that K is a <u>constant</u>. We may rewrite the equations (4) as follows:

$$\frac{\partial u}{\partial x} = K \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = - K \frac{\partial v}{\partial x} \qquad (4')$$

where K is <u>constant</u>.

Since K is constant, it is, of course, independent of x and y, and if it be computed at any point x_0, y_0 we should obtain its value in general. Let C_1 and C_2 be two curves given respectively by the equations $u = c_1$ and $v = c_2$ (Fig. 5), where c_1 and c_2 are constants. Considering here, as we do u and v to be conjugate functions, it follows from our definition (Cf. § 1) that if the curve given by the equation $u = c_1$ represents the lines of equipotential, then the curve given by the equation $v = c_2$ represents the corresponding lines of force; and conversely, if $v = c_2$ represents the lines of equipotential then $u = c_1$ represents the corresponding lines of force. It is thus seen that both $u = c_1$ and $v = c_2$ may be taken as curves representing

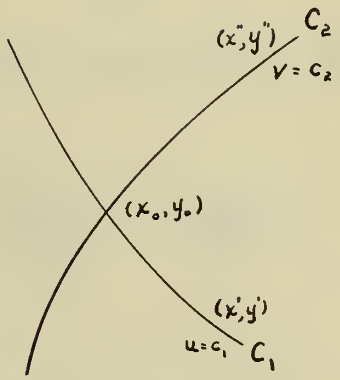

Fig. 5.

the lines of equipotential, and the reader will do well to bear it in mind in what follows.

Let x', y' denote any point on $u = c_1$. Then at any point of c_1 the components of the force against which work is done to produce the potential $u(x', y')$ are $-\frac{\partial u}{\partial x}$, $-\frac{\partial u}{\partial y}$. The magnitude of the intensity at any point (x', y') is then $\sqrt{(\frac{\partial u}{\partial x'})^2 + (\frac{\partial u}{\partial y'})^2}$. Likewise, the magnitude of the intensity of the force producing the potential $v(x'', y'')$ where x'', y'' are the coordinates of any point on c_2, is $\sqrt{(\frac{\partial v}{\partial x''})^2 + (\frac{\partial v}{\partial y''})^2}$. The point x_o, y_o is on both curves and by the definition of conjugate function the intensity of the force producing the two potentials is the same at any point in the field. Hence we have

$$\sqrt{(\frac{\partial u}{\partial x_o})^2 + (\frac{\partial u}{\partial y_o})^2} = \sqrt{(\frac{\partial v}{\partial x_o})^2 + (\frac{\partial v}{\partial y_o})^2} \tag{11}$$

But, from (4') we have for all points (x, y) exterior to the attracting mass

$$\sqrt{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2} = \pm K \sqrt{(\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2}$$

Consequently, for (x_o, y_o) we have

$$\sqrt{(\frac{\partial u}{\partial x_o})^2 + (\frac{\partial u}{\partial y_o})^2} = \pm K \sqrt{(\frac{\partial v}{\partial x_o})^2 + (\frac{\partial v}{\partial y_o})^2} \tag{12}$$

By comparing (11) and (12), we have

$$K = \pm 1 .$$

The direction of the intensity of the field is always along the tangent to the lines of force (or stream lines in case of velocity potential). In the two systems of equipotential curves under consideration the lines of force are every-

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$$[illegible] \qquad (11)$$

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$$[illegible] \qquad (12)$$

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where orthogonal. If we take the positive value of K_1, namely + 1, we have the Cauchy-Riemann differential equations written in the usual form.

We have thus proved that conjugate functions, as defined by us, satisfy the Cauchy-Riemann differential equations, i.e. the usually given definition of conjugate functions is a necessary consequence of our definition.[1]

In treatises on the theory of the potential it is shown that to every logarithmic potential function corresponds a conjugate function[2], which can be easily found. If one of two conjugate functions be given, it can be shown that the other is thereby determined, except for an additive constant. For, let for example u be given. Then

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \ ,$$

or from the Cauchy-Riemann differential equations, it follows that

$$dv = - \frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \ ,$$

and therefore

$$v = \int (- \frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy) + c \ .$$

1. It might be noted here that two functions U and V are often said to be conjugate if they satisfy the Cauchy-Riemann differential equations. This is the usual definition of conjugate functions. From this definition it follows that the curves given by $u=c_1$ and $v=c_2$ cut orthogonally and they may be thus considered as lines of equipotential and lines of force i.e. $u=c_1$ may be considered as the lines of potential, then $v=c_2$ becomes the corresponding lines of force and vice versa. Thus, it is seen that here orthogonality follows from the usual definition. In defining conjugate functions, we took this consequence of the usual definition as our definition, and we showed that the usual definition follows as a consequence of ours.

2. See Plemelj's Potentialtheoretische Untersuchungen, p. 3.

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We now proceed to prove a necessary and sufficient condition that two functions be conjugate.

THEOREM. <u>A necessary and sufficient condition that two functions u(x,y) and v(x,y) be conjugate is that the Cauchy-Riemann Differential Equations be satisfied.</u>

PROOF.

1) <u>The condition is necessary</u>. That this condition is necessary is seen at once. For, if u and v are two conjugate functions, then as we have already shown above, the Cauchy-Riemann differential equations are satisfied.

2) <u>The condition is sufficient</u>. To prove that the condition is also sufficient we have given the Cauchy-Riemann differential equations are satisfied, and we are to show that u and v are conjugate functions.

Since u and v satisfy the Cauchy-Riemann differential equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Multiplying these equations member by member we obtain

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0,$$

which is the condition of orthogonality.

Moreover, since u, v may each be regarded as a potential function and hence the derivatives $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 v}{\partial x^2}$, $\frac{\partial^2 v}{\partial y^2}$, exist, it follows immediately from the Cauchy-Riemann differential equations that by differentiation we have

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 .$$

Hence, both u and v satisfy Laplace's equation, and either u or v may be taken as a potential function. Furthermore, the condition of orthogonality clearly indicates that the curves $u = c_1$ and $v = c_2$ cut everywhere orthogonally, and thus if $u = c_1$ be taken as the lines of equipotential, $v = c_2$ will become the lines of force, and conversely, if $v = c_2$ be taken as the lines of equipotential $u = c_1$ becomes the lines of force. Hence, according to our definition of conjugate functions, u and v are conjugate and the condition is also sufficient.[1]

---

1. Since we are here leaving the discussion of conjugate functions, it may not be amiss to draw the reader's attention to a property of conjugate functions which is not generally made mention of in text-books in which the subject of conjugate functions is treated. The property consists in that <u>two conjugate functions say u and v are linearly independent</u>.

Indeed, if u and v are conjugate functions, then as we have seen they satisfy the Cauchy-Riemann Equations, i.e. we have $\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right\}$. Now if u and v were <u>linearly dependent</u>, then we would have an equation

$$c_1 u + c_2 v \equiv \text{constant}$$

where c_1, and c_2 are constants (not both zero). Differentiating the last equation first with respect to x and then with respect to y we get

$$c_1 \frac{\partial u}{\partial x} + c_2 \frac{\partial v}{\partial x} \equiv 0, \qquad c_1 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial y} \equiv 0 ,$$

whence by means of the derived relations (C-R equations) we obtain the following contradictory results

$$\frac{\partial u}{\partial x} \equiv 0 \qquad \frac{\partial u}{\partial y} \equiv 0$$

Hence u and v are <u>not</u> linearly dependent, and they are therefore linearly independent. (Compare: Klein, On Riemann's Theory of Functions. English translations by F. Hardcastle (1893) p. 38.)

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5. Condition that $u + iv$ be expressed as a function of z.

We shall now show that *if u and v are conjugate functions, then $u + iv$ can be expressed as a function of z*, where $z = x + iy$, i.e., $u + iv = f(z) \equiv f(x+iy)$. Since u and v are conjugate functions, therefore, they satisfy the Cauchy-Riemann differential equations (§ 4), and we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Putting $w = u + iv$, and differentiating partially with respect to x and with respect to y respectively, we get:

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \tag{1}$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \tag{2}$$

Now, by the Cauchy-Riemann differential equations (2) may be rewritten as follows

$$\frac{\partial w}{\partial y} = -\frac{\partial v}{\partial x} + i\,\frac{\partial u}{\partial x}$$

$$\frac{\partial w}{\partial y} = i^2\frac{\partial v}{\partial x} + i\,\frac{\partial u}{\partial x} \tag{3}$$

$$\frac{w}{y} = i\left(i\,\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)$$

Observing that the right-hand member (in the parenthesis) of (3) is identially equal to $\frac{\partial w}{\partial x}$ (see (1)), we have

$$\frac{\partial w}{\partial y} = i\,\frac{\partial w}{\partial x} \tag{4}$$

The last equation (4) is a linear partial differential equation. Letting $\frac{\partial w}{\partial x} = p$, and $\frac{\partial w}{\partial y} = q$, equation (4) becomes

$$ip - q = 0 ; \tag{5}$$

i.e. it is an equation of the type

$$Pp + Qq = R\ .[1]$$

Solving (5) by the ordinary method given for the solution of linear partial differential equations, we get

$$w = c_1 \qquad \text{and} \qquad x + iy = c_2$$

where c_1 and c_2 are constants.

Hence, it follows that

$$w = f(x + iy) ,$$

where f is an arbitrary function.

Replacing w by u + iv, and (x + iy) by z, we have

$$u + iv = f(z) .$$

<u>Another proof</u>[2]. Since u and v are conjugate functions, therefore according to § 4, each one of them satisfies Laplace's equation for two dimensions. We may then write

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 .$$

This equation has a general solution in finite terms, namely

$$v = F(x + iy) + \phi(x + iy) ,$$

where F and ϕ are arbitrary functions.

In order that v be wholly real, ϕ must be the function obtained from F when we change i into -i. Let F(x+iy) be equal

1. See A.R.Forsyth. Differential Equations. (Third Edition) § 185, pp. 354-355.

2. This proof is essentially the one given by Jeans. See Jeans Electricity & Magnetism, (Second edition), § 306, p. 261.

to $u' + iv'$, where u' and v' are real, then $\phi(x + iy)$ must be equal to $u' - iv'$, and thus we must have $v = 2u'$. If we introduce now a second function u equal to $-2v'$, we have

$$\begin{aligned} u + iv &= -2v' + 2iu' \\ &= 2\,i(u' + iv') \\ &= 2\,if(x + iy) \\ &= \psi\,(x + iy)\ , \end{aligned}$$

where $\psi(x + iy)$ is a general function of the single variable $(x + iy)$. As before putting $x + iy \equiv z$, we have

$$u + iv = \psi(z)\ .$$

6. <u>Definition of Function Holomorphic in a Given Region</u>.

Before we give the definition of a function holomorphic in a given region, we define a few terms which will be made use of in this thesis.

First of all then, we define a <u>region as a continuum of inner points</u>. It follows from this definition that a region does <u>not</u> include its boundary points. When the boundary of the region is included we speak of it as being a <u>closed region</u>.

We say that $z \equiv \alpha$ is a <u>regular point</u> of $f(z)$ if the given function $f(z)$ has a uniquely determined derivative at the point α and at every point in the neighborhood of α.

A point in every deleted neighborhood of which there are regular points, but which is itself <u>not</u> a regular point, is said to be a <u>singular point</u> of the given function.

<u>Definition</u>. A single valued function $f(z)$ is said to be holomorphic in a given region S, if for all points of S,

$$f(z) = u + iv\ ,$$

where u and v are conjugate functions in the sense defined by us. (Chapter II, § 1.)[1].

7. <u>Existence of a derivative of $(u + iv) \equiv f(z)$ with respect to z. A necessary and sufficient condition.</u>

Let $w = u + iv \equiv f(z)$ be a given function of z, and let z be a variable point in the neighborhood of z_0. Also let

$$\Delta z \equiv z - z_0 \,, \quad \text{and} \quad \Delta w \equiv f(z) - f(z_0) \equiv f(z_0 + \Delta z) - f(z_0) \,.$$

Now, if the expression

$$\frac{\Delta w}{\Delta z} \equiv \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

has a finite determinant limit which remains the same under all possible suppositions as to the path of approach of z to z_0, then this limit is called the "derivative" of the function f(z) at the point z_0, thus

$$\lim_{\Delta z = 0} \frac{\Delta w}{\Delta z} \equiv \lim_{\Delta z = 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \equiv f'(z_0) \quad ,$$

or for any point z,

$$f'(z) \equiv \lim_{\Delta z = 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \,.$$ [2]

We will now consider the circumstances under which a function $w = u + iv$ may have a derivative at the point $z = x + iy$.

Let z be given a real increment, then x is changed into $x + \Delta x$, while y remains unaltered, so that $\Delta z = \Delta x$, and we have

---

1. What we call a <u>function holomorphic in a given region</u> is by some authors called <u>holomorphic function</u>. Although the former term is somewhat longer than the latter, we prefer to use it and will do so throughout this treatment.
2. As in the calculus of real variables the symbols $D_z w$, $\frac{dw}{dz}$ are also frequently used here.

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u}{i\Delta y} + \frac{\Delta v}{\Delta y} \quad .$$

The existence of the derivative, however, as we have seen, involves the condition that $\frac{\Delta w}{\Delta z}$ shall give the same limiting value independent of the way in which Δz approaches zero. Moreover, since u and v are potential functions therefore the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist. Consequently, we must have

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = -\,i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad .$$

Equating the real and imaginary parts, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \qquad \frac{\partial u}{\partial y} = -\,\frac{\partial v}{\partial x} \text{ , i.e.}$$

the Cauchy-Riemann Differential Equations, which are a necessary condition for the existence of a derivative.

We will now show that this condition is also sufficient,[1] i.e. if u and v satisfy the Cauchy-Riemann differential equations, then u + iv has a derivative at the point z.

Indeed, let the increment of the independent variable be entirely arbitrary i.e. we make no supposition as to the relative magnitudes of its real and imaginary parts. Then the differential of the function is

$$du + idv = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right)dx + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right)dy \quad .$$

Hence we have

$$\frac{du + idv}{dx + idy} = \frac{\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right)dy}{dx + idy}$$

or

1. For a detailed discussion the reader is referred to an article by E. Goursat in Vol. I, p. 14 of the Transactions of the American Mathematical Society.

$$\frac{du + idv}{dx + idy} = \frac{\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right)\frac{\partial y}{\partial x}}{\left(1 + i\,\frac{dy}{dx}\right)} ,$$

In general this expression depends essentially upon $\frac{dy}{dx}$ in the limit; it will be independent of $\frac{dy}{dx}$ when and only when the term free from $\frac{dy}{dx}$ in the numerator as well as in the denominator has the ratio 1 : i to the coefficient of $\frac{dy}{dx}$, in other words, when

$$\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) .$$

But this equation is true when and only when the Cauchy-Riemann differential equations are here satisfied. Since, however,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\,\frac{\partial v}{\partial x} ,$$

the Cauchy-Riemann differential equations are here satisfied by hypothesis it follows that this expression, which by virtue of the condition stated in the hypothesis (satisfying C-R equations) is equal to either member of the equation

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = -\,i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} ,$$

is independent of $\frac{dy}{dx}$, or, what is the same thing, of the direction of approach to the point z.

We have thus proved that the existence of a derivative of the function $u + iv \equiv f(z)$ depends only on the existence of the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ satisfying the Cauchy-Riemann differential equations.

CHAPTER III.

Singularities. Fundamental Theorems.

Potential Continuation.

1. Singularities.

In treating singularities we shall confine ourselves to isolated singular points only. By an isolated singular point of a potential function we shall understand a point at which the function fails to possess some one or more of the properties of a potential function, although it has all of these properties at every other point in the neighborhood of this point.

In what follows we shall exclude removable discontinuities, considering singular points of this character as having been removed. What we understand by a removable discontinuity may be explained as follows.

In the definition of continuity of a function $u(x,y)$ at a point (a,b) are implied the following three characteristic properties:

1. $u(x,y)$ is defined at (a,b).
2. $\lim\limits_{\substack{x \doteq a \\ y \doteq b}} u(x,y)$ must exist at (a,b).
3. $\lim\limits_{\substack{x \doteq a \\ y \doteq b}} u(x,y) = u(a,b)$ at (a,b).

When a function possesses the first two of these properties and does not possess the third one, the function is of course

discontinuous at (a,b), but in this case the discontinuity is said to be removable.[1]

Considering then only isolated singular points of potential functions, we notice that "Sources" and "Sinks" are singularities of this kind. A "source" is a point from which arise lines of flow or lines of force. Moreover, the lines of flow or the lines of force have uniform intensity as to magnitude. It is in this sense that we employ the term "source" in the succeeding discussion. A negative source is called a "sink". If the total flux outwards across a small closed surface surrounding the point be m, then m is called the <u>strength</u> of the source.[2]

Suppose we have a source at each of the points α_1, α_2, α_3, ..., α_n having the strength k_1, k_2, k_3, ..., k_n respectively, and let there be a sink at each of the points β_1, β_2, β_3, .., β_m of strength λ_1, λ_2, λ_3, ..., λ_m respectively. We have then

$$w \equiv u+iv = \log \frac{(z-\alpha_1)^{k_1}(z-\alpha_2)^{k_2}\ldots(z-\alpha_n)^{k_n}}{(z-\beta_1)^{\lambda_1}(z-\beta_2)^{\lambda_2}\ldots(z-\beta_m)^{\lambda_m}} \quad [3] \qquad (1)$$

Let

$$z - \alpha_1 \equiv \rho_1(\cos\theta_1 + i\sin\theta_1) ,$$

$$z - \alpha_2 \equiv \rho_2(\cos\theta_2 + i\sin\theta_2) ,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$z - \alpha_n \equiv \rho_n(\cos\theta_n + i\sin\theta_n) .$$

Also let

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1. For illustrations of removable discontinuities the reader is referred to Burkhardt-Rasor, Theory of Functions of a Complex Variable, Examples 8 and 9, pp. 252-253.

2. See: Lamb - Hydrodynamics (Third Edition), p. 55.
3. See: Professor E.J.Townsend's Notes on the Theory of Functions of a Complex Variable, p. 173.

$$z - \beta_1 \equiv r_1(\cos\phi_1 + i \sin\phi_1) ,$$
$$z - \beta_2 \equiv r_2(\cos\phi_2 + i \sin\phi_2) ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$z - \beta_m \equiv r_m(\cos\phi_m + i \sin\phi_m) .$$

Making these substitutions, (1) becomes

$$w \equiv u + iv = \log \left(\frac{\rho_1^{k_1}\rho_2^{k_2}\cdots\rho_n^{k_n}}{r_1^{1} r_2^{2}\cdots r_m^{m}}\right) +$$

$$i\Big[(k_1\theta_1 + k_2\theta_2 + \cdots + k_n\theta_n) - (\lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_m\phi_m)\Big] .$$

Apparently, when we have a source only ($z = \alpha_1 = \alpha_2 = \ldots \alpha_n$), we get

$$w = \log 0 = -\infty :$$

and if we have a sink only ($z = \beta_1 = \beta_2 = \ldots \beta_m$), we obtain

$$w = \log \infty = \infty .$$

Consequently, "sources" and "sinks" are singular points.

Now, if at a fixed point P in a thin plane plate, Fig. 6, there is a sink of strength numerically equal to m, and at another point P' in the same plate is a source of equal strength, and if P' be made to approach P in such a manner that the product m.PP' remains constant, say equal to n, then we say that we have a "plane doublet" of strength n, the axis of which is Px, the limiting position of the straight line drawn from P to P'. A "plane doublet" is also called a pole. A "plane quadruplet" is formed of two equal and opposite plane doublets in the same manner that a doublet is formed of a source and an equal sink. We may use the term

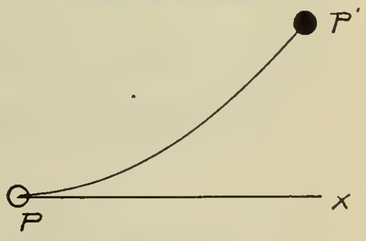

Fig. 6.

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"motor" to denote in general a source, a sink, a doublet, a quadruplet, or any other combination of sources or sinks at a single point.[1]

Consider now the streaming given by the real part of the function

$$w = -a \log \frac{z - z_1}{z - z_0},$$

where a is real and $a > 0$, i.e. consider

$$u = -a \log \frac{r}{r'},$$

where

$$r = |z - z_1| \quad , \qquad r' = |z - z_0| \quad .$$

Apparently, we may consider here as having a source at z_1 and a sink at z_0 of equal strength. Suppose now that we bring the points z_1 and z_0 together. If P is an arbitrary fixed point of the plane other than z_0 or z_1, then when we bring z_0 and z_1 together, lim. $\frac{r}{r'} = 1$, and if a remains constant we have

$$\lim u = -a \log \lim_{r \doteq r'} \frac{r}{r'} = -a \log 1 = 0 \; .$$

In order to avoid this possibility, we allow at the same time a to become infinity; i.e. we allow the strength of the source and sink to become greater and greater. The necessity of this measure is obvious from physical considerations, since most of the heat will stream directly over from the source to the sink.

The same conclusion can be reached analytically as follows:

1. Peirce. Newtonian Potential Function. pp. 434-435.

Let

$$z_0 = \text{const.} \quad z_1 = z_0 + he^{\alpha i}, \quad a = \frac{A}{h} ,$$

where $A > 0$.

Then we have

$$-a \log \frac{z-z_1}{z-z_0} = \frac{A}{h}\left[-\log\left(\frac{z-z_0-he^{\alpha i}}{z-z_0}\right)\right] = \frac{A}{h}\left[-\log\left(1-\frac{he^{\alpha i}}{z-z_0}\right)\right] =$$

$$= \frac{Ae^{\alpha i}}{z-z_0} + \frac{1}{2}\,\frac{Ae^{2\alpha i}}{(z-z_0)^2}\,h + \frac{Ae^{3\alpha i}}{(z-z_0)^3}\,h^2 + \ldots$$

Consequently,

$$\lim_{z_1 = z_0}\left[-\,a \log \frac{z-z_1}{z-z_0}\right] = \frac{Ae^{\alpha i}}{z-z_0} \quad .$$

The value of A represents the strength of the "plane doublet" or the order of the pole, while α gives the direction of the axis of the "plane doublet".

We may also remark that the function u becomes infinite at a source or a sink; on the other hand in the neighborhood of a pole the function assumes every value.

Consider now two poles of the same order whose axes are the lines passing through these poles and having opposite directions. Suppose for the present that these poles are the north and south poles of a sphere while their axes extend along the meridian $\phi = 0$. Because of the symmetry of a sphere, the streaming takes place in the lower hemisphere precisely in the same manner as it does in the upper hemisphere. In particular

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the equator will be a streaming line. Moreover, along the meridians $\phi = 0, \pi$, the streaming takes place, going into the poles along one of these lines and going out along the other.

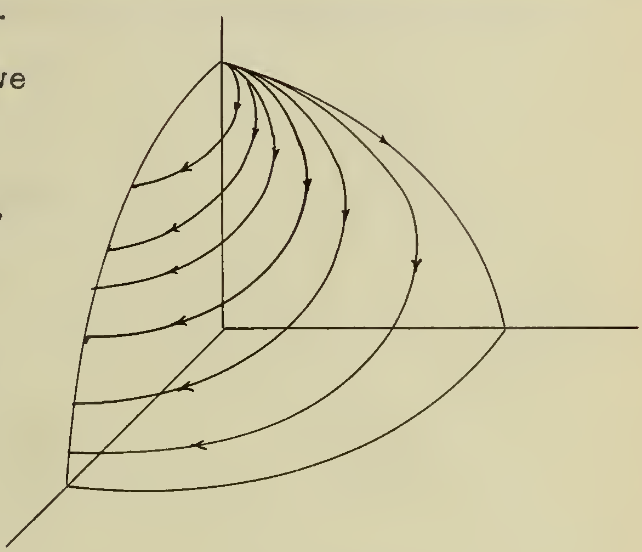

Fig. 7

Projecting the sphere stereographically upon the plane, we have

$w = a(\frac{1}{z}+z+c)$, where $a > o$,

and c is a constant.

$$w \equiv u + iv$$
$$= a(\frac{1}{z}+z+c)$$
$$= a(\frac{1}{x+iy}+x+iy+c)$$
$$= a(\frac{x-iy}{x^2+y^2}+ x+iy+c)$$
$$= \frac{ax}{x^2+y^2} + ax + ac + i(\frac{-ay}{x^2+y^2} + ay) .$$

Equating th real parts and the imaginary parts, we get

$$u = \frac{ax}{x^2+y^2} + ax + ac ,$$

$$v = \frac{-ay}{x^2+y^2} + ay .$$

It is easily seen by actual substitution into the above expressions for u and v that the axes of reals $y = 0$ as well as the unit circle $x^2+y^2 = 1$ are stream lines.

In a half of the upper hemisphere a closed streaming takes place which repeats itself in the remaining three lunes through a reflection upon the two limiting planes, that is the

one through the equator and the one through the meridian $\phi = 0$. Two points in which the four lines are joined deserve particular attention. In the neighborhood of one of these points the streaming takes place in the directions indicated in Fig. 8, while in the neighborhood of the other point the direction is reversed. This is apparently only possible when the velocity at these points becomes zero. We speak of each of these points as a "point of zero velocity".

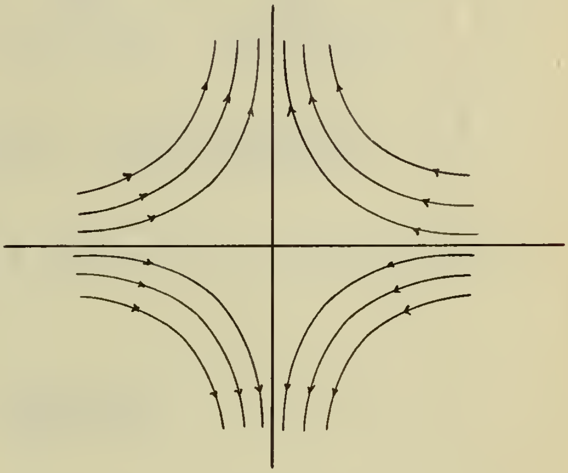

Fig. 8

Consider now a linear transformation of the plane

$$z = \lambda \frac{z' - h}{z' + h}, \qquad \lambda > 0$$

whereby the points $z = 0, \infty$ will be carried over respectively into the points $z' = h, -h$ $(h > 0)$; by doing so the sense in which the axes through the pole $z = 0$ was taken remains unaltered. For simplicity we may put $\lambda = 1$, and then the transformation becomes

$$z = \frac{z' - h}{z' + h} .$$

Substituting this value of z in the foregoing expression for w, and putting $c = -2$, we get:

$$w = a\left(\frac{1}{z} + z + c\right)$$
$$= a\left(\frac{z'+h}{z'-h} + \frac{z'-h}{z'+h} - z\right)$$

$$= a\left[\frac{(z'+h)^2 + (z'-h)^2 - 2(z'^2-h^2)}{(z'-h)(z'+h)}\right]$$

$$= a\left(\frac{2h}{z'-h} - \frac{2h}{z'+h}\right), \tag{2}$$

$$w \equiv u + iv = a\left[\frac{2h}{x'+iy'-h} - \frac{2h}{x'+iy'+h}\right]$$

$$= a\left[\frac{2h(x'-h-iy')}{(x'-h)^2+y^2} - \frac{2h(x'+h-iy')}{(x'+h)^2+y'^2}\right]$$

Equating the real and the imaginary parts, and dropping the accents, we obtain

$$u = \frac{2ah(x-h)}{(x-h)^2+y^2} - \frac{2ah(x+h)}{(x+h)^2+y^2},$$

$$v = \frac{-2ahy}{(x-h)^2+y^2} + \frac{2ahy}{(x+h)^2+y^2}.$$

By substitution it will be seen that these equations are satisfied by $x = 0$, $y = 0$, and hence these lines are among the stream lines. Now the points of zero velocity are at $z = 0, \infty$. We now let both poles come together. The strength of the plane doublets under consideration will thereby increase constantly, or what is the same the order of the pole obtained by bringing two poles together will increase. Indeed, dropping the accents in (2) and reducing the right-hand member to a common denominator, we get

$$w = \frac{4h^2a}{z^2-h^2},$$

which remains also true is we put $4h^2a$ = const., say A.

Passing to the limit as $h \doteq 0$, we have

$$\lim_{h \doteq 0} w = \frac{A}{z^2} \; .$$

Thus we see that by bringing together two simple poles in accordance with the conditions here considered we obtain a pole of the second order.

It can be proved that in general by bringing together n simple poles we obtain a pole of the n^{th} order.[1]

We have now seen how singular points may arise in the potential function u(x,y). Corresponding to these singularities we have singular points of the analytic function f(z) = u+iv. If f(z) is single-valued, then we may have both poles and essential singular points just as when the properties of these functions are developed from the purely mathematical point of view. If these points constitute a closed boundary, then we speak of that curve as the natural boundary of the analytic function.

2. Green's Theorem.

Green's theorem may be stated as follows:

THEOREM. In a given finite region S let C be the complete boundary of any portion of the plane such that C lies within S and incloses only points of S. If in a given region u(x,y) and v(x,y) are continuous real functions of x and y together, having the continuous partial derivatives $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, then

$$\int_C u dx + v dy = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx dy \; ,$$

1. Osgood. loc. cit. p. 617. Aufgabe 3.

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where the double integral is to be taken over the region bounded by C.

Proof. It is seen at once from the characteristic properties of potential functions (Cf.§ 6, Chapt. I) that they satisfy all the conditions required by the theorem. Indeed, potential functions are continuous real functions of x and y together which follows from the total differentiability of potentials (§ 6, Chapt. I, (7)). Furthermore, the partial derivatives $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ of two potential functions u and v are continuous (§ 6, Chapt. I, (2)). Hence all the conditions set forth in the theorem are fulfilled and thus Green's theorem holds for potential functions.

It will be seen that potential functions possess even more properties than the theorem requires. For example, they have the property that the second partial derivatives exist and also are finite.

3. *The Cauchy-Goursat Theorem.*

THEOREM. *Let f(z) be holomorphic in a given finite region S and let C be the complete boundary of any portion S', of the plane such that C lies within the given region S and incloses only points of S; then*

$$\int_C f(z)dz = 0 .$$

Let $f(z) = u + iv$, $z \equiv x + iy$. Since u and v are potential functions, the first partial derivatives of u and v are continuous functions of x and y together. Moreover, since f(z) is by hypothesis holomorphic in S, it follows from the definition of a function holomorphic in a given region (Chapt. II, § 6) that

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u and v are conjugate functions. We have already proved however (Chapt. II, § 4) that conjugate functions satisfy the Cauchy-Riemann differential equations. Consequently u and v satisfy the above equations, that is, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} .$$

On the other hand we have

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx) . \tag{1}$$

Each of the integrals on the right-hand side od (1) is a line integral. Moreover, it is known that u and v are continuous functions of x and y together and possess continuous partial derivatives

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y} .$$

We may therefore apply the theorem stating the necessary and sufficient condition that a line integral shall vanish, namely:

If in a given region $P(x,y)$ and $Q(x,y)$ are continuous real functions of x and y together, having the continuous partial derivatives $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$, then the necessary and sufficient condition that the integral

$$\int Pdx + Qdy$$

shall vanish is that we have

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} .$$

for all points of the given region S.[1]

1. See Professor E.J.Townsend's Notes on the Theory of Functions of a Complex Variable. p. 63, Th. II.

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Applying this theorem to the integrals in the right-hand member of (1), we have as the necessary and sufficient condition that these integrals vanish the partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which are the Cauchy-Riemann differential equations. But as we have already seen u and v satisfy the Cauchy-Riemann differential equations, hence

$$\int_C (udx - vdy) = 0 \quad \text{and} \quad \int_C (udy + vdx) = 0 .$$

Making these substitutions in (1), it follows at once that

$$\int_C f(z)dz = 0 .$$

SECOND METHOD.[1]

Let $f(z) = u + iv$ and $z = x + iy$. Since $f(z)$ has a continuous derivative in S, the first partial derivatives of u and v are continuous functions of x and y together, and for the same reasons as given in the first method, u and v also satisfy the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} .$$

Now, we have

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (udy + vdx) \tag{1}$$

Applying Stoke's theorem[2] to each of the integrals on the right-hand side of (1), we get

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1. This method is essentially the one given in Pierpont's Theory of Functions of a Complex Variable. § 90, p. 183.
2. Loc. cit. § 80, p. 158.

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$$\int_C udx - vdy = -\int_{C'}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy = 0 \ ,$$

$$\int_C udy + vdx = \int_{C'}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy = 0 \ .$$

Hence (1) becomes

$$\int_C f(z)dz = 0 \ .$$

From the foregoing theorem we have the following corollary:

<u>Corollary</u>. Let $f(z)$ be holomorphic in the given region S and let C_1, C_2 be any two ordinary curves joining the points A and B of S, where C_1, C_2 lie wholly in S and enclose only points of S. Then we have

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \ .$$

Indeed, since $C_1\bar{C}_2$[1] is a closed curve, therefore by the Cauchy-Goursat theorem we have

$$\int_{C_1\bar{C}_2} f(z)dz = 0 \ .$$

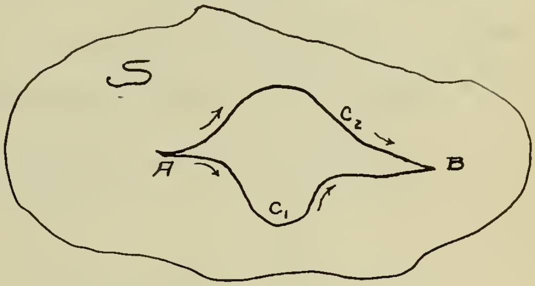

Fig. 9

But,

$$\int_{C_1\bar{C}_2} f(z)dz = \int_{C_1} f(z)dz + \int_{\bar{C}_2} f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0 \ .$$

Hence,

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \ ,$$

1. The dash on C_2 designates that C_2 is taken in the <u>negative</u> direction.

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i.e. the value of the integral $\int f(z)dz$ is <u>independent</u> of the path along which the integral is taken.

4. Green's Function.[1]

A given function "g" is called Green's function if in a given region S it satisfies the following conditions:

1) g is a single-valued potential function at all points of the region S, with the exception of a point O.

2) g has a logarithmic infinity at the point O, that is

$$g = \log \frac{1}{r} + w(x,y) ,$$

where r is the distance from the point O[2], and w(x,y) is a potential at the point O.

3) g vanishes along the boundary of S.

In the language of electrical phenomena, Green's function is a potential due to a positive unit of electricity placed at the point O together with that of the charge which it induces on the surface S made conducting and connected to earth[3].

5. Cauchy's Integral Formula.

We have already remarked (§ 5, Chapt. I) that when the points at which a potential function is considered lie outside of the attractive mass, the function may be called either a potential function or a harmonic function. Moreover, it is apparent that if such is the case, the potential and harmonic

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1. The name "Green's Function" is due to C. Neumann.

2. The point "O" is by some authors called a "Pole". See Webster, Electricity and Magnetism, p. 390.

3. For a fuller discussion of the significance of Green's function in physics see Osgood, Lehrbuch der Funktionentheorie, 2d Edition, vol. 1, pp. 631-2.

functions must, under the given conditions, possess the same properties. It is known from the theory of harmonic functions that if the value of the harmonic function is known on the boundary of a region S, then the value of the function at any inner point of S is uniquely determined.[1] From what we have said the same must be true of potential functions, i.e. there is a theorem in the potential theory analogous with the well known Cauchy-Integral formula of the function theory. We shall now give two proofs of this theorem for potential functions.[2]

THEOREM. <u>Given a finite closed region S whose boundary C consists of a finite number of ordinary curves. If f(z) is holomorphic in S, then for any inner point z of S we have</u>

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} ,$$

<u>the integral being taken positively around the whole boundary of the region.</u>

FIRST METHOD

In the theory of harmonic functions the following extremely important expression for a harmonic function u(x,y) is to be found

$$u(x,y) = \frac{1}{2\pi}\int_C U(t)\,\frac{\partial g}{\partial n}\,ds \;, \quad (1)$$ [3]

where U(t) is the value of u on the boundary, and g is Green's function. From what we have already said, it follows that if

1. G. Kowaleski, Die Komplexen Veranderlichen und Ihre Funktionen § 62.
2. A third proof will be found in Harkness and Morley's Introduction to Analytic Functions. § 170, pp. 324-326.
3. Osgood, Lehrbuch der Funktionentheorie, 2d Ed. vol. 1, p. 632 (4), also p. 629, Satz.

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u(x,y) is a potential function, then we should have

$$u(x,y) = \frac{1}{2\pi} \int_C U(t) \frac{\partial g}{\partial n} ds , \tag{2}$$

where U(t) is the value of u on the boundary and g is Green's function. Similarly, if v(x,y) is another potential function, we have

$$v(x,y) = \frac{1}{2\pi} \int_C V(t) \frac{\partial g}{\partial n} ds . \tag{3}$$

Myltiplying (3) by i and adding the result to (2) we have, since $f(z) = u + iv$,

$$f(z) \equiv u+iv = \frac{1}{2\pi}\int_C \left[U(t)+iV(t)\right] \frac{\partial g}{\partial n} ds . \tag{4}$$

Since $U(t) + iV(t)$ is a function of t, we may give to t the values of the complex variable z along the curve C and write $U(t) + iV(t) \equiv f(t)$ for these values. Hence, we have from (4)

$$f(z) = \frac{1}{2\pi} \int_C f(t) \frac{\partial g}{\partial n} ds . \tag{5}$$

Now let h be the conjugate function to the Green function g. We may then write

$$g + ih = - \log(t - z) + P(t,z) ,$$ [1]

where, for a fixed value of z, P is holomorphic for all values of t on the boundary of the closed region S.

The functions g and h are conjugate functions and g is the Green function. Consideraing g + ih as a function of s, n, we have

1. Osgood, Lehrbuch der Funktionentheorie, 2^{d} Ed. vol. 1, p. 678.

$$\frac{\partial(g + ih)}{\partial t} = \frac{\partial(g + ih)}{\partial s}\frac{ds}{dt} \tag{6}$$

Now

$$\theta = \tau - 90^\circ, \quad d\theta = d\tau \quad , \quad ds = \rho d\theta = \rho d\tau \quad .$$

The variable τ is complex, putting then

$$\tau = \rho e^{i\theta} = \rho e^{i(\tau - 90^\circ)} \ ,$$

we find

$$\begin{aligned} d\tau &= \rho e^{i(\tau - 90^\circ)} . \, id \\ &= i\rho d\tau \cos(\tau - 90^\circ) + i \sin(\tau - 90^\circ) \\ &= i\rho d\tau(\sin \tau - i \cos \tau) \\ &= \rho d\tau (\cos \tau + i \sin \tau) \\ &= ds . e^{i\tau} \end{aligned} \tag{7}$$

From (6) and (7) we have

$$\frac{\partial(g + ih)}{\partial t} = \frac{\partial(g + ih)}{\partial s} e^{-i\tau} \ .$$

that is

$$\frac{\partial g}{\partial s} . e^{-\tau i} + i\frac{\partial h}{\partial s} . e^{-\tau i} = \frac{\partial(g + hi)}{\partial t}$$

Since $\frac{\partial g}{\partial s} = 0$ [1], the last equality becomes

$$i \frac{\partial h}{\partial s} e^{-\tau i} = \frac{\partial(g + hi)}{\partial t} \ ,$$

or

$$\frac{\partial h}{\partial s} e^{-\tau i} = - i \frac{\partial(g + hi)}{\partial t} \ ,$$

whence

$$- \frac{\partial h}{\partial s} e^{-\tau i} dt = i \frac{\partial(g + hi)}{\partial t} dt \ , \tag{8}$$

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1. This follows from § 4, (3) .

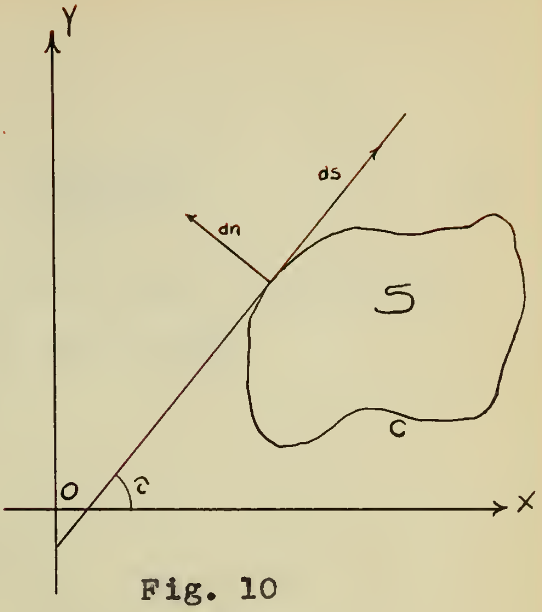

Fig. 10

Now

$$\frac{\partial(g+hi)}{\partial t} = -\frac{1}{t-z} + \frac{\partial P(t,z)}{\partial t}, \quad (8)$$

$$i\frac{\partial(g+hi)}{\partial t}dt = -\frac{idt}{t-z} + i\frac{\partial P(t,z)}{\partial t}dt$$

$$= i\,\frac{\partial P(t,z)}{\partial t}dt - \frac{dt}{t-z} \qquad (9)$$

Since g and h are conjugate functions, consequently the Cauchy-Riemann differential equations are satisfied and we have

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial g}, \qquad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$$

But

$$\frac{\partial g}{\partial y}\cos(90^\circ-\theta) = -\frac{\partial h}{\partial x}\cos(90^\circ-\theta),$$

Consequently

$$\frac{\partial g}{\partial n} = -\frac{\partial h}{\partial s},$$

and hence

$$\frac{\partial g}{\partial n}ds = -\frac{\partial h}{\partial s}ds,$$

or since

$$ds = dt.e^{-i\tau} \quad [1]$$

it follows that

$$\frac{\partial g}{\partial n}ds = -\frac{\partial h}{\partial s}dte^{-i\tau} \qquad (10)$$

Combining (8), (9), and (10), we get

$$\frac{\partial g}{\partial n}ds = i\left[\frac{\partial P(t,z)}{\partial t}dt - \frac{dt}{t-z}\right],$$

1. (7) of this article.

Hence (5) may be rewritten as

$$f(z) = \frac{i}{2\pi}\left[\int_C f(t)\frac{\partial P(t,z)}{\partial t}dt - \int_C \frac{f(t)dt}{t-z}\right] . \tag{11}$$

However, by the Cauchy-Goursat theorem, the first integral in the right-hand member of (11) vanishes, and we get

$$f(z) = -\frac{i}{2\pi}\int_C \frac{f(t)dt}{t-z} ,$$

or

$$f(z) = -\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} .$$

SECOND METHOD[1]

Let S be a region bounded by an ordinary curve C. Let x and y be the rectangular coordinates of a point M lying in S. Consider now the complex variable

$$z = x + iy ,$$

let f(z) be a function which is holomorphic in S. We may then put

$$f(z) = U + iT ,$$

where U and T are conjugate functions. We have then from the definition of conjugate functions (Chapt. II, § 1)

$$\Delta U = 0 ,$$

$$\Delta T = 0 ,$$

for all points of the region S. On the other hand U and T satisfy the Cauchy-Riemann differential equations

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial T}{\partial y} , \\ \frac{\partial U}{\partial y} &= -\frac{\partial T}{\partial x} \end{aligned} \tag{1}$$

---

1. The proof given here is essentially the one given by H. Poincaré. Theorie du Potentiel Newtonien, (Paris, 1899), §70, pp. 149-53

[illegible]

$$[illegible]$$

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Now, let M' be another point of the region S, and let x',y' be its rectangular coordinates, and z' the value of z at this point. We have then

$$z' = x' + iy' .$$

Consider now the function

$$\log \frac{1}{z - z'} .$$

We may put

$$\log \frac{1}{z - z'} = V + iW ,$$

where V stands for $\log(\frac{1}{r})$, and W, as well as V, is a function of the real variables x, y when we assume M' to be a fixed point. Now, let U' and T' be the values of the potential functions U and T respectively at the point M'. From the relation[1]

$$U' = \frac{1}{2\pi} \int \left[\log(\frac{1}{r}) \cdot \frac{dU}{dn} - U \frac{d \log(\frac{1}{r})}{dn} \right] ds$$

upon replacing $\log(\frac{1}{r})$ by V, we obtain

$$2\pi U' = \int (V \frac{dU}{dn} - U \frac{dV}{dn}) ds$$

$$2\pi T' = \int (V \frac{dT}{dn} - T \frac{dV}{dn}) ds \qquad (2)$$

Transform the given axes of coordinates to new axes which are respectively parallel to the original ones, the new origin being the point (x_0, y_0). Turn the new axes through an angle α, and call these new axes of coordinates ξ, η .

1. See H. Poincaree loc. cit. § 69, pp. 148-149.

The formulas of transformation from the original pair of rectangular axes to the new are:

$$x = \xi \cos \alpha - \eta \sin \alpha + x_0 ,$$
$$y = \xi \sin \alpha + \eta \cos \alpha + y_0 .$$

The Cauchy-Riemann differential equations (1) may be now written

$$\frac{\partial U}{\partial \xi} = \frac{\partial T}{\partial \eta} ,$$

$$\frac{\partial U}{\partial \eta} = - \frac{\partial T}{\partial \xi} .$$

Suppose for example that the new origin has been transformed to the point A lying on the curve C, and that the new axes coincide with the tangent and normal at this point, (Fig. 11). We have now

$$\frac{dU}{ds} = \frac{dT}{dn} ,$$

$$\frac{dU}{dn} = - \frac{dT}{ds} ,$$

the positive direction of the tangent being the direction of inverse rotation (counter-clockwise). Considering the clockwise direction as positive we have

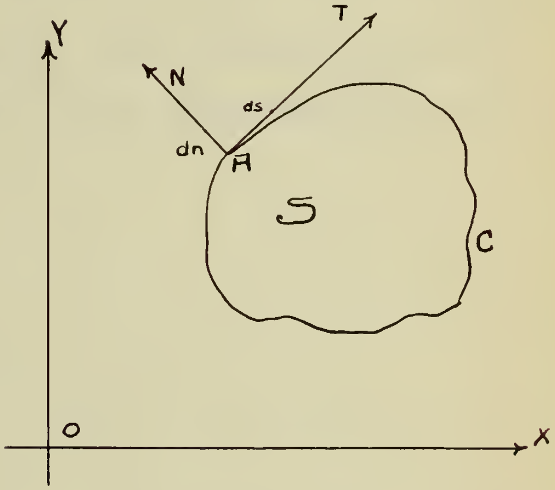

Fig. 11.

$$- \frac{dU}{ds} = \frac{dT}{dn} ,$$

$$\frac{dU}{dn} = \frac{dT}{ds} .$$

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Similarly, we have

$$-\frac{dV}{ds} = \frac{dW}{dn} ,$$

$$\frac{dV}{dn} = \frac{dW}{ds} .$$

The integrands of (2) may now be simplified, and (2) may be rewritten thus

$$2\pi U' = \int_C (VdT - UdW) ,$$
$$2\pi T' = - \int_C (VdU + TdW) . \qquad (3)$$

These two curvilinear integrals are taken in a positive direction, that is in a counter-clockwise direction. The function W is not a single-valued function, but U, V and T are single-valued, and we have

$$\int VdT = - \int TdV ,$$

$$\int VdU = - \int UdV .$$

Now the integrals in (3) become

$$2\pi U' = - \int TdV + UdW ,$$
$$2\pi T' = \int UdV - TdW . \qquad (4)$$

On the other hand we have:

$$\frac{dz}{z - z'} = dV + idW ,$$

and therefore

$$\int f(z) . \frac{dz}{z-z'} = \int (U + iT)(-dV - idW),$$

or

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$$\int f(z)\frac{dz}{z-z'} = \int -UdV + TdW - i\int TdV + UdW$$

$$= -2\pi T' + i2\pi U'^{1}$$

$$= 2\pi i(U' + iT')$$

$$= 2\pi i f(z') \ .$$

Hence,

$$f(z') = \frac{1}{2\pi i}\int \frac{f(z)dz}{z-z'} \ .$$

Note. Put

$$z' = z \ ,$$

$$z = t \ ,$$

then we get the Cauchy-Integral formula in the form in which it is generally given,

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z} \ .$$

6. Consequences of the Cauchy Integral Formula.

In the preceeding article we derived the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z} \ .$$

If $z = z_o$, where z_o is a given point of S, then we have

$$f(z_o) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z} \ .$$

This formula indicates that the values of a function f(z) which is holomorphic in a finite closed region S are fully determined for values within S is we know its values upon the boundary of S.

1. This follows by replacing the integrals on the right-hand side by their values found in (4).

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Moreover, differentiating (1) with respect to z_0, the following important formulas, expressing the successive derivatives of a function holomorphic in a given region at a given finite point are obtained:

$$f'(z_0) = \frac{1!}{2\pi i}\int_C \frac{f(t)dt}{(t-z_0)^2} ,$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(t)dt}{(t-z_0)^3} ,$$

$$f'''(z_0) = \frac{3!}{2\pi i}\int_C \frac{f(t)dt}{(t-z_0)^4} ,$$

. . . .

. . . .

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(t)dt}{(t-z_0)^{n+1}} .$$

The integrals in the right-hand sides of these equations are all finite and determinate for every position of z_0 within C. We may thus conclude that every function holomorphic in a given region admits of an infinite number of successive derivatives at every interior point of the region in which the function is holomorphic. Moreover, each of these derivatives is holomorphic and hence continuous.

7. Expansion of a Function Holomorphic in a Given Region in a Power Series.

Having established the Cauchy-integral formula, we are in a position now to prove the following very important

THEOREM. *If f(z) is holomorphic in a given region S, then in the neighborhood of any point z_0 in S, f(z) can be represented by a power series in $(z-z_0)$, and that in one and*

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<u>only</u> <u>one</u> <u>way</u>, <u>namely</u>

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + f''(z_0)\,\frac{(z-z_0)^2}{2!}$$

$$+ f^n(z_0)\,\frac{(z-z_0)^n}{n!} + \ldots\ldots\ldots$$

Let z_0 be <u>any</u> point in the given region S, and let z_0 be the center of a circle C lying within S; moreover let (z_0+t) be any inner point of the circle C (Fig. 12), and z' any variable point in C. From the Cauchy integral formula it follows that

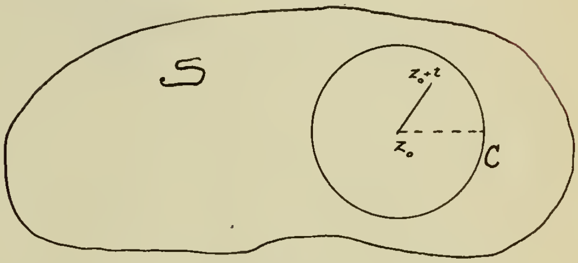

Fig. 12

$$f(z_0+t) = \frac{1}{2\pi i}\int_C \frac{f(z')dz'}{z'-(z_0+t)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z')dz'}{z'-z_0-t}$$

$$= \frac{1}{2\pi i}\int_C f(z')dz'\left[\frac{1}{z'-z_0} + \frac{t}{(z'-z_0)^2} + \ldots + \ldots\right.$$

$$\left. + \frac{t^n}{(z'-z_0)^{n+1}} + \frac{t^{n+1}}{(z'-z_0)^{n+1}(z'-z_0-t)}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z')dz'}{(z'-z_0)} + t\,\frac{1}{2\pi i}\int_C \frac{f(z')dz'}{(z'-z_0)^2} + \ldots +$$

$$+ t^n\,\frac{1}{2\pi i}\int_C \frac{f(z')dz'}{(z'-z_0)^{n+1}} + R_n \qquad (1)$$

where

$$R_n = \frac{1}{2\pi i}\int_C f(z')dz'\,\frac{t^{n+1}}{(z'-z_0)^{n+1}(z'-z_0-t)} \qquad (2)$$

However, it follows from § 6 of this chapter, that

[illegible]

[illegible]

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$$\frac{1}{2\pi i}\int_C \frac{f(z')dz'}{z'-z_0} = f(z_0) ,$$

$$\frac{1}{2\pi i}\int_C \frac{f(z')dz'}{(z'-z_0)^2} = f'(z_0) ,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{1}{2\pi i}\int_C \frac{f(z')dz'}{(z'-z_0)^{n+1}} = \frac{f^n(z_0)}{n!} .$$

Hence, (1) may be rewritten as follows:

$$f(z_0+t) = f(z_0) + tf'(z_0) + t^2\frac{f''(z_0)}{2!} + \dots +$$
$$+ t^n\frac{f^n(z_0)}{n!} + R_n . \tag{3}$$

Now from (2) we find that

$$|R_n| \leqq \frac{1}{2\pi}|t|^{n+1}\int_C \left|\frac{f(z')}{(z'-z_0)^{n+1}(z'-z_0-t)}\right||ds| .$$

Let $\left|\frac{f(z')}{(z'-z_0)^{n+1}(z'-z_0-t)}\right|$ take its maximum value at the point α on C. Then we have

$$|R_n| \leqq \left|\frac{t}{\alpha-z_0}\right|^{n+1}\left|\frac{f(\alpha)}{\alpha-z_0-t}\right| .$$

Since, however, $\left|\frac{t}{\alpha-z_0}\right| < 1$, it follows that $|R_n|$ diminishes as n increases, and, when n is chosen sufficiently great, can be made less than any assignable quantity. It follows then that R_n has zero for its limit when $n = \infty$, and (3) becomes

$$f(z_0+t) = f(z_0) + tf'(z_0) + t^2\frac{f''(z_0)}{2!} + \dots +$$
$$+ t^n\frac{f^n(z_0)}{n!} + \dots\dots$$

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Writing now z for z_0+t, this series becomes the well known form of Taylor's Series,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + f''(z_0)\frac{(z-z_0)^2}{2!} + \ldots . +$$
$$+ f^n(z_0)\frac{(z-z_0)^n}{n!} + \ldots . .$$

and hence the theorem is established.[1]

8. Potential Continuation.

We have thus far considered potential functions in a given region and have shown how such functions are related to functions of a complex variable which are holomorphic in this region. We have not inquired into the question as to how large that region might be in any given case. The object of this article is to show that the region within which a given function is known to be a potential function can in many cases be enlarged, and to give a method by which this process of enlarging the given region may be accomplished. It is seen at once that the problem undertaken here is analogous to that considered in the theory of functions of a complex variable, namely analytic continuation.

Let u_1 be a potential function defined for all points of a region S_1. Suppose it is possible to find a second potential function u_2 defined for all points of a region S_2 overlapping the region S_1. (Fig. 13). Moreover, let

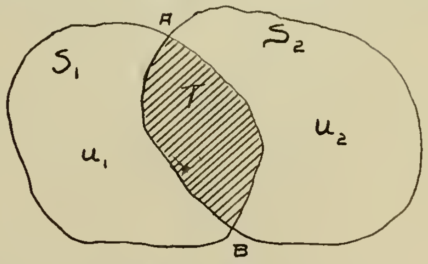

Fig. 13

1. For the proof of the uniqueness of the representation see Prof. E.J.Townsend's Notes on the Functions of a Comples Variable, pp. 307-308.

$u_1 = u_2$ in the part common to both regions, end points A, B at most excepted, then the functions u_1 and u_2 define a new function u which is a potential function in the enlarged region S composed of S_1 and S_2, such that $u = u_1$ in S_1 and $u = u_2$ in S_2. In T we have $u = u_1 = u_2$. The function u_2 we then call a <u>potential continuation</u> of u_1, and the process of finding such a function we call the process of <u>potential continuation</u>.

The question naturally arises here whether or not the process of potential continuation is unique; that is, whether there exists one and only one function u_2 as defined above. We shall be in position to answer this question after we prove a certain lemma. In order, however, to be able to prove this lemma we must first prove the following theorem.

THEOREM. <u>Let u be a potential function defined for a given region S and let U_1 be its mean value along the circumference of a circle C_1, lying wholly within S and inclosing only points of S. Also let C_2 be a circle concentric with C_1 and lying wholly within S and inclosing only points of S, and let the mean value of u along the circumference of C_2 be U_2. Then we have</u>

$$\int_0^{2\pi} U_1 d\phi = \int_0^{2\pi} U_2 d\phi \; .$$

<u>Proof</u>. Let u_0 be the value of u at the center of the concentric circles C_1 and C_2, that is at (x_0, y_0).

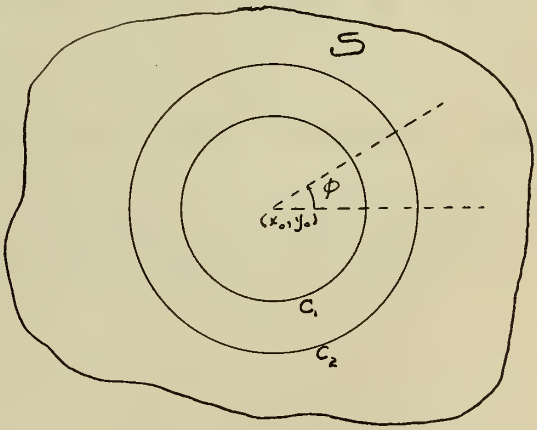

Fig. 14

From the theorem of the Mean Value[1], it follows that when we consider the circle C_1 ,

$$u_o = \frac{1}{2\pi}\int_0^{2\pi} U_1 d\phi \quad ,$$

where U_1 represents the values of u upon the circle C_1. Again, considering the circle C_2, we have

$$u_o = \frac{1}{2\pi}\int_0^{2\pi} U_2 d\phi \ .$$

Hence,

$$\int_0^{2\pi} U_1 d\phi = \int_0^{2\pi} U_2 d\phi \ ,$$

as the theorem requires.

We are now in position to prove the following lemma.

LEMMA. Let u be a potential function defined for a given region S. If u vanishes for all points of a region T lying within S, then u vanishes (in general) for all points of S.

Let "α" be a point lying within the region T, and "β" an arbitrarily choses point of the region S, (Fig. 15). Let α and β be connected by an ordinary curve L lying wholly within S. We will now cover L with a finite number of circles C_1, C_2, ..., C_{k-1}, C_k, lying wholly within S and in such a manner that the center of each succeeding circle lies within the circle immediately preceeding it, i.e. the center of the circle C_k lies within the circle C_{k-1}. Moreover, we may take α to be the center of C_1.

---

1. Osgood, Loc. cit., p. 621, Der Mittelwertsatz.

The validity of the foregoing lemma depends upon the existence of a certain integral known as Poisson's integral just as the Taylor series depends upon the Cauchy integral. We know that in the theory of harmonic functions[1] it is shown that if u be a function harmonic in a given region S, and if O be a point within S, (Fig. 15), then u can be expanded in the following series.

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta) \ , \tag{1}$$

where

$$a_n = \frac{1}{\pi a^n}\int_0^{2\pi} U \cos n\phi d\phi \ , \quad b_n = \frac{1}{\pi a^n}\int_0^{2\pi} U \sin n\phi d\phi . \tag{2}$$

This series converges and represents the function u at all points of the largest circle K that can be drawn about O as a center lying wholly within S not including any boundary points of S (Fig. 16). Moreover, this representation is unique,[2] and since the series (1) converges uniformly it can be integrated term by term. It can also be differentiated term by term.

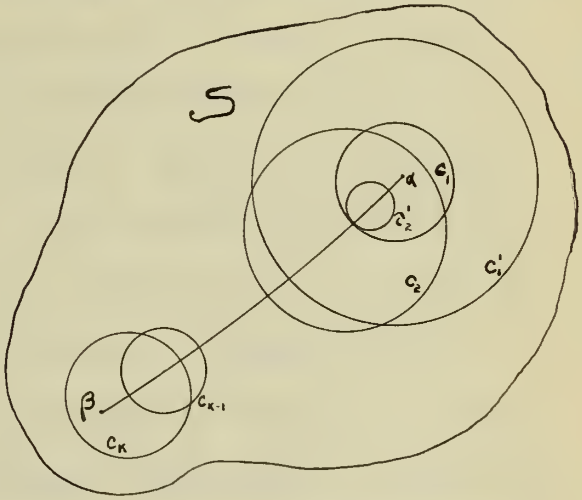

Fig. 15

Now, let C_1 be any circle drawn about α as a center and lying wholly within T. Also let C_1' be the largest concentric

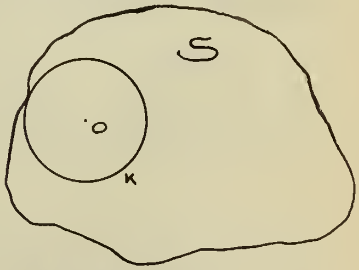

Fig. 16

1. Chapter I, § 5.
2. Osgood, Lehrbuch der Funktionentheorie, I, p. 656 et seq.

[illegible]

$$[illegible] \quad (\,)$$

$$[illegible] \quad (\,)$$

[illegible]

circle with C_1 and lying wholly within S. The coefficients of the expansion (1) are given by the integrals (2), the integrands of which are potential functions. We may thus apply the theorem proved in this article to the circles C_1 and C_1'; and since the circle C_1 lies wholly within T and by hypothesis u vanishes for all points of T, consequently the expansion (1) vanishes identically in T for each coefficient vanishes within T. Hence, by the theorem proved in this article, each coefficient of (1) vanishes within and upon C_1' and consequently (1) vanishes identically within and upon C_1'. About α_2, the center of the circle C_2 draw a concentric circle C_2' lying wholly within C_1', or by drawing C_2' sufficiently small it can be made to lie within the region T. We know already that each coefficient of (1) will vanish within and upon C_2' since at most it will lie within C_1' for each point of which (2) vanishes, and consequently (1) will vanish identically within and upon C_2'. However, C_2 is a circle concentric with C_2' and lies wholly within S. Hence, by the theorem referred to above, each coefficient of (1) will vanish within and upon C_2 and thus (1) vanishes identically within and upon C_2. Proceeding in this manner, it is possible at least theoretically, to obtain after a finite number of operations a circle C_k having β as an inner point and for all points of which (1) vanishes identically. Hence u vanishes at β . However, β is any point of S, and therefore u vanishes for all points of S, as stated in the lemma.

We are now in position to prove that the function u_2 as defined above is uniquely determined. Indeed, suppose that another potential continuation of u_1, say u_2', could be found.

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Then u_1 and u_2' would define a new function, say u', which is a potential function in the enlarged region composed of the regions S_1 and S_2, and moreover $u' = u_1 = u_2'$ in T. (Fig. 13).

We have

$$u - u_1 = 0 \text{ , in the region } T \text{ ,}$$

but $u' - u_1 = 0$, " " " " .

Consequently $u' - u = 0$ " " " "

and by the lemma

$$u' - u = 0 \text{ in } (S_1 + S_2)$$

i.e. $u' = u$ in the enlarged region composed of S_1 and S_2. Hence the uniqueness of u follows.

It is evident that u_1 likewise may be regarded as a potential continuation of u_2. Instead of overlapping, the two regions S_1 and S_2 may have an arc C of an ordinary curve as a boundary between the two regions. In this case we imagine an infinitesimal region lying on both sides of the arc C, and in which the functions u_1 and u_2 satisfy the same conditions as in the case of two overlapping regions. In either case we speak of the functions u_1 and u_2 as <u>elements of the function u</u>. The element from which the other elements are obtained by the process of potential continuation is called the <u>primitive element</u>, and the remaining elements become continuations from it.

We have defined what we mean by "potential continuation" and we have shown that the process of "potential continuation" is unique. We shall now develop a method by which the process of "potential continuation" can be actually carried out.

However, before doing so, it will be necessary for us to consider Poisson's Integral.

9. Poisson's Integral.

Let O and O' be conjugate points with respect to a given circle, and let δ and δ' be the distances of a variable point P(ξ, η) on the circle from O and O' respectively.

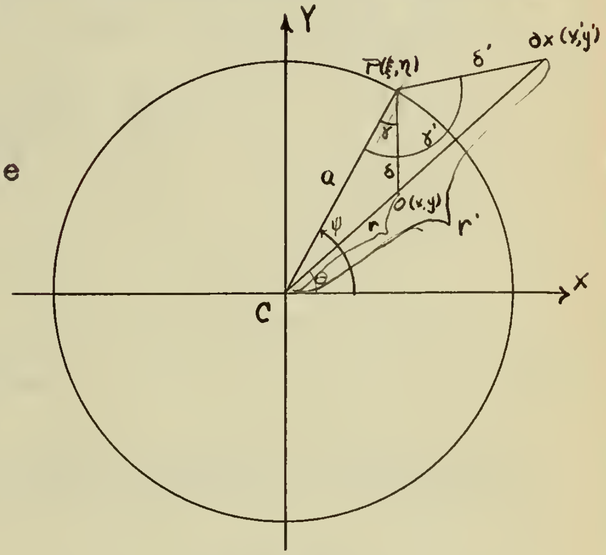

Fig. 17

The equation of the circle may be written as

$$\frac{\delta}{\delta'} = \text{const.} = k ,$$

where k depends only upon the co-ordinates x, y but not upon ξ, η, and

$$g = -\log\delta + \log\delta' + \log k$$

is then the Green function of the circular sheet because it vanishes along the circumference. It is a potential function owing to the fact that the logarithm of a distance from a point, is a solution of Laplace's equation, and becomes infinite at o as ~~a~~-log δ. Now we have

$$\frac{\partial g}{\partial n} = -\frac{\frac{\partial\delta}{\partial n}}{\delta} + \frac{\frac{\partial\delta'}{\partial n}}{\delta'} = \frac{\cos\gamma}{\delta} - \frac{\cos\gamma'}{\delta'} \qquad (1)$$

Moreover, we have

$$r^2 = a^2 + \delta^2 - 2a\,\delta\cos\gamma ,$$

$$r'^2 = a^2 + \delta'^2 - 2a\,\delta'\cos\gamma' ,$$

where r = CO, and r' = CO'.

Whence we find

$$\cos\gamma = \frac{a^2 - r^2 + \delta^2}{2a\delta} = \frac{a^2 - r^2}{2a\delta} + \frac{\delta}{2a},$$

$$\cos\gamma' = \frac{a^2 - r'^2 + \delta'^2}{2a\delta'} = \frac{a^2 - r'^2}{2a\delta'} + \frac{\delta'}{2a}.$$

Substituting these values of $\cos\gamma$ and $\cos\gamma'$ in (1) we get

$$\frac{\partial g}{\partial n} = \frac{a^2 - r^2}{2a\delta^2} + \frac{1}{2a} - \frac{a^2 - r'^2}{2a\delta'^2} - \frac{1}{2a} =$$

$$= \frac{a^2 - r^2}{2a\delta^2} - \frac{a^2 - r^2}{2a\delta'^2} =$$

$$= \frac{1}{2a}\left[\frac{a^2 - r^2}{\delta^2} - \frac{a^2 - r'^2}{\delta'^2}\right] \qquad (2)$$

Since O and O' are conjugate points with respect to the circle of radius a, it follows that

$$rr' = a^2 \text{ [1]}$$

or

$$r' = \frac{a^2}{r},$$

and moreover,

$$\frac{\delta}{\delta'} = \frac{a - r}{r' - a} = \frac{r}{a} = k,$$

whence

$$\delta' = \frac{\delta a}{r}.$$

The second member in the parenthesis of (2) can now be transformed as follows

$$\frac{a^2 - r'^2}{\delta'^2} = \frac{a^2 - \frac{a^4}{r^2}}{\frac{\delta^2 a^2}{r^2}} = \frac{r^2 - a^2}{\delta^2}$$

1. See Professor E.J.Townsend's Loc. cit. p. 212.

It is seen at once that (2) now becomes

$$\frac{\partial g}{\partial n} = \frac{1}{a}\left(\frac{a^2 - r^2}{\delta^2}\right) \tag{3}$$

Since, however,

$$\delta^2 = a^2 + r^2 - 2ar\cos(\theta - \phi) ,$$

$$= a^2 - 2ar\cos(\theta - \phi) + r^2 ,$$

Making this substitution in (3) we get

$$\frac{\partial g}{\partial n} = \frac{1}{a}\,\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi)+r^2} . \tag{4}$$

For the particular case of a circle, formula (1) of § 5 (Chapt. I) becomes

$$u = \frac{a}{2\pi}\int_0^{2\pi} U\,\frac{\partial g}{\partial n}\,d\phi ,$$

and substituting here the value of $\frac{\partial g}{\partial n}$ from (4), we get

$$u = \frac{1}{2\pi}\int_0^{2\pi} U\,\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi)+r^2}\,d\phi .$$

This integral is the so-called Poisson's integral.

Now we are in position to consider a method by which the process of potential continuation can be carried out.

10. The Symmetric Method of Potential Continuation.

Let u be a potential function defined for points of a region S_1 lying in the upper half plane and having a segment AB of the X-axis as a part of its boundary. Moreover, let u vanish along some segment of the X-axis. Also if $P_1(x_1,y_1)$ be any inner point of the region S_1, and $P_2(x_2,y_2)$ a point in the

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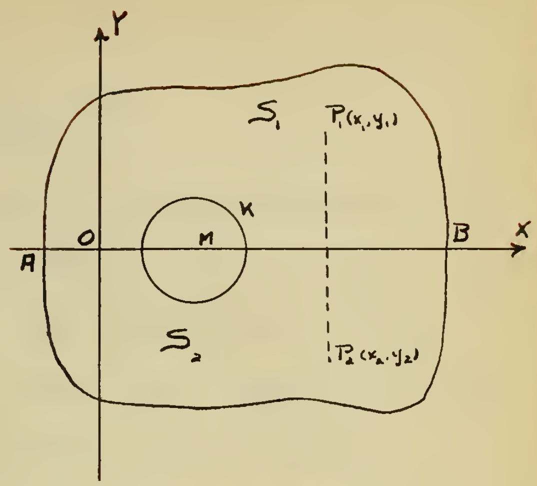

Fig. 18

lower half plane symmetrical to P_1 relative to the X-axis, then $x_2 = x_1$, $y_2 = - y_1$. Now in the region S_2 lying in the lower half plane and symmetrical to S_1 with respect to AB, we define the function u as follows: the value of u at P_2 shall be equal to the value which u assumes at P_1 with the sign changed, that is

$$u(x_2,y_2) = - u(x_1,y_1) .$$

The function u thus defined is apparently continuous in the extended region,[1] possesses partial derivatives at all inner points of the region S_2, and in particular we have

$$\left.\frac{\partial u}{\partial x}\right]_{\text{at } P_1} = - \left.\frac{\partial u}{\partial x}\right]_{\text{at } P_2}, \quad \left.\frac{\partial u}{\partial y}\right]_{\text{at } P_1} = \left.\frac{\partial u}{\partial y}\right]_{\text{at } P_2} \qquad (1)$$

and moreover, u satisfies Laplace's equation at least at the points P_1 and P_2. Whether u is a potential for points along the X-axis is still a question to be answered since we do not know whether $\frac{\partial u}{\partial y}$ is in general exists for those points. The answer to the question is found in a theorem due to Schwarz.[2]

Let M be any arbitrary point of the X-axis which lies within the extended region, and furthermore let K be a circle described about M as a center and lying within the same region. Then Schwarz applies Poisson's integral (see § 9) to the circle K.

1. Cf. Prof. E.J.Townsend's Notes on the Functions of a Complex Variable, p. 323.
2. Schwarz, Berliner Berichte, 1870, p. 744. Also Werke, vol. 2, pp. 149-151.

$$\frac{1}{2\pi}\int_0^{2\pi} U \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\psi)+r^2}\, d\psi \quad ,$$

where U designates the values of u along the circumference of K. Poisson's integral represents in K a potential function u_1, which first of all is identical with u along the upper part of the circumference.[1] We now maintain that u and u_1 are likewise identical for points along the X-axis. In fact θ (in Poisson's integral) has the value 0 or π for each point of the X-axis, and hence it follows that the second factor of the integrand assumes equal values for two points upon the circumference of K which are symmetrical with respect to the X-axis, while the first factor assumes equal and opposite in sign values at the same points. Consequently the integrand vanishes. It is thus seen that u and u_1 are two functions which are potential functions in the regions bounded by the upper and lower semi-circles, and moreover they assume equal values along the boundaries of these semi-circles. Now we may apply a well known theorem of harmonic functions, and since in our discussion harmonic and potential functions are identical, we may cite a theorem of potential function which states:

<u>If u_1 and u_2 are two potential functions defined for a given region T and if u_1 and u_2 assume the same boundary values everywhere along the boundary of T, then $u_1 = u_2$ for every inner point of T.</u>[2]

Applying this theorem to the case at hand, we conclude that u and u_1 assume the same values for all points within the

1. Osgood, Loc. cit. p. 640, Satz. 3.
2. Osgood, Loc. cit. p. 624, Satz. 4.

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semi-circles and herewith also for all points of the circle K. Since, however, u_1 is a potential at M, therefore u must be a potential at M. However, M is any point of the X-axis lying within the extended region, and hence u is a potential for all points of the X-axis lying in the same region. The only possible restriction is thus removed and the proof is now completed.

As a result of the above discussion we can now state the following theorem.

THEOREM. If u is a potential function for all points of a given region S having as a part of its boundary a segment C of a straight line, and if u vanishes everywhere along C, then u can be continued potentially over C. Moreover, the potential continuation of the given function attains at a point P_2 which is symmetrical to P_1 with respect to C, a value equal and opposite in sign to that possessed by the given function at P_1, while the conjugate function assumes equal values at P_1 and P_2.

It should be possible to develop a method of potential continuation by means of power series. The advantage of the Schwarz method of potential continuation, however, is the ease with which the continuation can be obtained. All that is needed is to reflect the given region upon the X-axis and associate with the reflected region a function possessing the property

$$u(x_2,y_2) = - u(x_1,y_1) .$$

In the case where AB is not a segment of the axis of reals, it can be made so by a proper transformation of the plane.[1]

1. See Osgood, Loc. cit. pp. 666-667.

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By the theorem on inversion the arc may be inverted into a line by taking the center of inversion at any point of the arc or the arc produced.[1] In this way the process of potential continuation is generalized and can be applied whether the continuation be taken with respect to a line or an arc of an ordinary curve.[2] For a fuller discussion of the generalized method of potential continuation as well as of some of the fundamental theorems dealing with potential continuation the reader is referred to Osgood's Lehrbuch der Funktionentheorie, pp. 666-673. What we have called "potential continuation" Osgood calls "harmonic continuation".

11. Analytic Function.

By the aid of the results of the preceeding articles concerning potential continuation, we can formulate now the definition of an analytic function. It should also be noted in this connection, that if u and v are potential functions, then $f(z) = u + iv$ is likewise a potential function and satisfies Laplace's equation. Now, if we know the values of a function and its derivatives at any point, then an element, say $\phi_1(z)$ of that function is uniquely determined. By the process of potential continuation we can extend the region in which the function is thus defined by determining other elements of the function and their corresponding regions. This extended region forms a connected region S within which a function is defined by its elements. If we suppose the region S to be extended as far as possible by means of potential continuation, then the corres-

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1,2. Wilson, Advanced Calculus, pp. 542-543.

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ponding aggregate of elements fully defines a function $F(z)$ in S such that $F(z)$ is equal to each of the elements $\phi(z)$ for those values of z for which $\phi(z)$ is defined, that is $F(z) \equiv \left\{\phi(z)\right\}$. The function $F(z)$ so defined is called an analytic function.

12. Conclusion.

We wish to sum up in this section the principal results of this thesis. Having given a brief summary of the general properties of potential functions, we entered upon the discussion of conjugate functions.

According to the definition generally given in the theory of functions of a complex variable, two functions $u(x,y)$ and $v(x,y)$ are said to be conjugate in a plane region S if throughout this region they are single-valued, continuous, have continuous partial derivatives and satisfy the Cauchy-Riemann differential equations. From this definition some important physical properties follow. We have taken for our definition of conjugate functions two of these properties (§ 1, Chapt. II) and we have shown how the customary definition of conjugate functions necessarily follows. Moreover, we have shown that the customary definition furnishes us with a necessary and sufficient condition that two functions be conjugate. Having done this, we are in position then to use the general theory of conjugate functions as treated in the theory of functions of a complex variable.

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We have next defined a function holomorphic in a given region in terms of potential functions and derived a necessary and sufficient condition that such a function possess a derivative.

The question of singularities necessarily claimed our attention next. We have shown that "sources" and "sinks" are singular points; that a "plane doublet" is a singular point corresponding to a "pole" in the function theory and that by bringing together two simple poles ("plane quadruplet") we obtain a pole of the second order, and in general by bringing together n simple poles we obtain a pole of the n^{th} order.

We have next shown that the fundamental theorems of the function theory, such as the Cauchy-Goursat theorem, the Cauchy integral formula hold likewise here and we have actually developed them.

We have also shown that potential functions can be expanded in a series of such a form that the usual power series development of functions which are holomorphic in a given region follows.

We have shown at last how the ordinary conception of a monogenic analytic function can be developed from fundamental notions of the potential function.

In making the fundamental conception in the development of the theory of analytic functions that of potential functions, in the sense in which we have used that term, we have made certain broader assumptions than is usually done in the usual mathematical

considerations of those functions. For example, our hypothesis assumes the continuity of the first derivative of the given function. It also assumes the existence of the second derivatives. These assumptions need not be made in the mathematical consideration of the properties of such functions. Those properties are, however, as is well known, direct consequences of the usual assumptions made. The purpose has been to show that the essential facts concerning analytic functions can be developed from the usual assumptions concerning the properties of potential functions rather than to develop a logical, detailed development of the theory as a whole. Having shown how these important properties can be developed, the complete development follows as a consequence. To complete the development is merely a matter of detail and does not concern the purpose of this thesis, which in brief is to show the possibility of developing the general theory of analytic functions from the fundamental conceptions of physics.

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